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ORMOND STONE W. E. BYERLY H. S. WHITE
W. F. OSGOOD MAXIME BÔCHER

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ON THREE DIMENSIONAL DETERMINANTS.*

By E. R. Hedrick.†

1. Introduction.

The extension of the idea of ordinary (or plane) determinants to arrays in three dimensions was first made by de Gasparis‡ in 1861, who defined a determinant of three dimensions, and gave some simple theorems, including the multiplication theorem. He was followed by Padova,§ Armenanti,‖ Garbieri,¶ Zehfuss,** and Dahlander;†† of whom the latter two probably worked independently of de Gasparis. None of these papers contain much new work, except that of Zehfuss, who gives the expressions for some elementary invariants. Later, several more important papers appeared, by Lloyd Tanner,‡‡ Scott,§§ Gegenbauer, ‖‖ and von Escherich.¶¶ Of these, the papers of Scott and Gegenbauer are most important. Scott deals mainly with determinants of special form; and Gegenbauer studies determinants of special form, and the application of the work to the invariant theory and to complex number systems. Both Gegenbauer and von Escherich have complicated their work greatly, and it would seem needlessly, by working in m dimensions at

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* This paper was read before the American Mathematical Society at the meeting of Feb. 25, 1899.

† A considerable portion of the material for this paper was prepared in collaboration with Mr. W. D. Cairns, now of Oberlin College, O.; but he should not be held responsible for any statements made in it.

‡ A brochure entitled "Sur les déterminants dont les éléments ont plusieurs indices." [I have not seen this brochure.] Also: *Rend. dell' accad. Napoli.*, VII, p. 118, 1868.

§ *Batt. Gior. di Mat.*, VI, p. 182.

‖ *Ibid.*, VI, p. 175.

¶ *Ibid.*, XV, p. 89.

** *Ueber eine Erweiterung des Begriffes der Determinanten.* Frankfurt a -M., 1868.

†† *Öfvers. af K. Vet. Akad. Förh.* Stockholm, 1863, p. 295.

‡‡ *Proc. Lond. Math. Soc.*, X, 1879, p. 167.

§§ *Ibid.*, XI, 1880, p. 17; *ibid.*, XIII, 1882, p. 33.

‖‖ *Wien. Akad. Denkschriften*, Bd. 43, 1881, p. 17; *ibid.*, Bd. 46, 1883, p. 291; *ibid.*, Bd. 50, 1885, p. 145; *ibid.*, Bd. 55, 1889, p. 39; *ibid.*, Bd. 57, 1890, p. 739. *Wien. Akad. Sitzungsberichte*, Bd. 101,2ᴬ, 1892, p. 425.

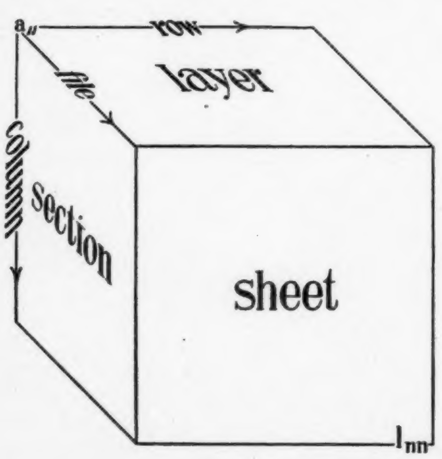
¶¶ *Wien. Akad. Denkschriften*, Bd. 43, 1881, p. 1.

once, in place of 3 dimensions. For while essentially new ideas are introduced in passing from two to three dimensions, the generalization past that point is perfectly obvious and presents nothing new.*

In this paper, it is proposed to take a somewhat new point of view, in basing the work on a new definition of the determinant. The work will be done for 3 dimensions only, thus securing the aid of a geometric picture. It is hoped that the simplicity thus gained will more than compensate for the slight labor of extending the theorems to the case of m dimensions. In the course of the paper several results will be stated, which are believed to be new.

2. Definition.

Let us think of n^3 elements placed in an array in a cube in a manner similar to the arrangement of n^2 elements in a square to form an ordinary determinant. There will be three mutually perpendicular lines, which we will call *rows*, *columns*, and *files*, respectively, if they are horizontal, vertical, perpendicular to the plane of the paper. There will be three mutually perpendicular planes, which we will call *layers*, *sections*, and *sheets*, respectively, if they are perpendicular to the *columns*, *rows*, *files*.



We now define† the value of the cubic‡ determinant to be the algebraic sum obtained by adding all possible determinants formed by taking any set of n columns, no two of which lie in the same section or sheet, and constructing a determinant of these columns, placing them in the order in which their

* Reference to the subject is made in the following, also. It is believed that these make the list fairly complete. (1) Waelsch, *Wien. Monatshefte*, IX, p. 213. (2) *Encyklopädie der Math. Wiss.*, Bd. 1, p. 45. (3) Günther, *Lehrbuch der Determinantentheorie*, p. 186 *et seq.* (4) Scott, *Theory of Determinants*, pp. 89–98. (5) Mansion, *Eléments de la theorie des déterminants*, Ed. 1883, p. 22 (note).

† This definition is strictly analogous to the definition of ordinary determinants in terms of products.

‡ So, in general, we should define a determinant of m dimensions (or of *class m*) to be equal to the sum of $n!$ determinants of class $(m-1)$.

respective sheets occur in the cubic determinant; and affixing the sign + or − according as the number of inversions in the order of the sections from which the columns came, is even or odd. There are $n!$ ordinary determinants in the expansion of a cubic determinant of order n. Thus for a determinant of the third order:

$$D \equiv \begin{array}{ccc} \boxed{\begin{matrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{matrix}} & & \\ & \boxed{\begin{matrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{matrix}} & \\ & & \boxed{\begin{matrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{matrix}} \end{array} \equiv \begin{array}{ccccc} a_{11} & a_{12} & a_{13} & & \\ a_{21} & b_{11} & b_{12} & b_{13} & \\ a_{31} & b_{21} & c_{11} & c_{12} & c_{13} \\ & b_{31} & c_{21} & c_{22} & c_{23} \\ & & c_{31} & c_{32} & c_{33} \end{array}$$

$$\equiv \begin{vmatrix} a_{11} & b_{12} & c_{13} \\ a_{21} & b_{22} & c_{23} \\ a_{31} & b_{32} & c_{33} \end{vmatrix} + \begin{vmatrix} a_{12} & b_{13} & c_{11} \\ a_{22} & b_{23} & c_{21} \\ a_{32} & b_{33} & c_{31} \end{vmatrix} + \begin{vmatrix} a_{13} & b_{11} & c_{12} \\ a_{23} & b_{21} & c_{22} \\ a_{33} & b_{31} & c_{32} \end{vmatrix} - \begin{vmatrix} a_{11} & b_{13} & c_{12} \\ a_{21} & b_{23} & c_{22} \\ a_{31} & b_{33} & c_{32} \end{vmatrix} - \begin{vmatrix} a_{12} & b_{11} & c_{13} \\ a_{22} & b_{21} & c_{23} \\ a_{32} & b_{31} & c_{33} \end{vmatrix} - \begin{vmatrix} a_{13} & b_{12} & c_{11} \\ a_{23} & b_{22} & c_{21} \\ a_{33} & b_{32} & c_{31} \end{vmatrix}.$$

It is seen from this expansion that the sign of any determinant in the expansion will be + or −, according as the number of inversions in it of the last subscript (in the above notation) is even or odd.

Also, the cubic determinant is seen to be equal to the sum of the *products* formed by taking the product of any n elements, no two of which lie in the same layer, section, or sheet and affixing to it the sign + or −, according as the sum of the number of inversions of the two subscripts, when the letters are arranged alphabetically, is even or odd.* There are $(n!)^2$ terms in this expansion.

It now follows that in the above definition we may interchange the words *layer* and *section*, and simultaneously the words *row* and *column*, for the final developments will be precisely as before. Thus, in the previous example:

$$D \equiv \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ b_{21} & b_{22} & b_{23} \\ c_{31} & c_{32} & c_{33} \end{vmatrix} - \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ b_{31} & b_{32} & b_{33} \\ c_{21} & c_{22} & c_{23} \end{vmatrix} + \textit{four more similar determinants.}$$

* This has been taken as a *definition* by other writers, and the above definition deduced as a theorem.

If we wish to develop by *files*, we shall find no development into ordinary determinants.* If we define a function, however, which is formed exactly like an ordinary determinant, except that each of its terms is positive:

$$\left\{\begin{matrix} a_1\, a_2 & . & . & . & a_n \\ b_1\, b_2 & . & . & . & b_n \\ . & . & . & . & . \\ . & . & . & . & . \\ q_1\, q_2 & . & . & . & q_n \end{matrix}\right\} \equiv \sum (a_1\, b_2 \; . \; . \; . \; . \; q_n)$$

we see at once that we may develop by files in terms of these *Scott's functions*,† as I shall call them. Form all the Scott's functions possible by taking for the rows any n files of the cubic determinant, no two of which lie in the same layer or section; affix to each the known sign of its principal diagonal; — the sum of these functions is the cubic determinant. For, it is seen that every term resulting from the complete expansion of the cubic determinant occurs (except perhaps for sign) in the complete expansion of this sum of Scott's functions. Moreover, the signs will be correct. For the sign of all terms resulting from the expansion of any one of the above Scott's functions, will be the same in the complete development of the cubic determinant.

3. Elementary Theorems.

In the following theorems, the proofs will be indicated only when some point is particularly interesting or difficult. In general the method of proof is to develop the cubic determinant into ordinary determinants, when the truth is at once seen. No effort has been made to render this list complete, for too many theorems suggest themselves at once by analogy from two dimensional determinants; and only those will be found here which are to be used in this paper. The first writer who has stated a theorem is indicated in parentheses at its close. The proofs in this paper are generally different from those given by others, as the theorems themselves, in all cases, were found independently. The writer regrets not having seen de Gasparis' first paper.

(1). In any cubic determinant, the sections may be interchanged with the layers, in order, without altering the value of the determinant. But the

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* This lack of symmetry is essential, and is noticeable throughout the work. It is due to the fact that the number of inversions of three series of numbers when grouped into sets of three each, one from each set, is not constant when the order of arrangement of the sets is changed.

† Such a function has been defined and used by F. R. Scott: *Proc. Lond. Math. Soc.*, Vol. XIII, p. 36.

sheets cannot so be interchanged either with the sections or with the layers. (Padova.) Proof by development into plane determinants, by Sec. 2.

(2). In any cubic determinant, if two adjacent sections be interchanged, the determinant changes sign. The same thing is true for layers. But any two sheets may be interchanged without any alteration of the determinant. (Dahlander.) Proof by development into plane determinants.

(3). If each of the elements of any layer (or section) be equal to the corresponding element of any other layer (or section), the determinant vanishes. But two (or any number) of the sheets may be identical, and the determinant will not necessarily vanish. (Dahlander.) Proof by development into plane determinants.

(4). If each of the elements of any plane (layer, section, or sheet) be multiplied by any quantity, the value of the determinant is multiplied by that quantity. (Gegenbauer, 1881.) Proof by development into plane determinants by Sec. 2; or into Scott's functions if we are dealing with sheets (and hence files). It is necessary to notice that if any row of a Scott's function be multiplied by any quantity, the function is multiplied by that quantity. This comes directly from the definition.

(5). The value of a cubic determinant in which all n sheets are equal, is $n\,!$ times the plane determinant formed from the elements in any one sheet, as they stand. (Dahlander.) Proof direct by development by columns, *e. g.*, by Sec. 2.

(6). The value of a cubic determinant in which, of the n sheets, the first q are equal to the first, while the remaining $(n-q)$ are equal to the $(q+1)$th identically, is $q\,!\,(n-q)\,!$ times the sum of the $n\,!/q\,!\,(n-q)\,!$ determinants formed by replacing in the 1st sheet, any $(n-q)$ columns (or rows) by the corresponding $(n-q)$ columns (or rows) of the $(q+1)$th sheet. (Scott, 1882.) Proof direct by development by columns (or rows) by Sec. 2. This theorem can easily be extended to the case where q_i sheets are equal to the ith, where $\Sigma\, q_i = n$.

(7). A cubic determinant in which any two sheets are composed of columns (or rows) such that every column (or every row) in each, is some multiple of a single column (or row) vanishes identically. For, each plane determinant in the development by columns (or rows) vanishes.

(8). Let us define the *principal diagonal* to be that diagonal of the cube which contains the elements *a_{11}, b_{22}, c_{33}, etc. Let us define the *diagonal plane*

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* See p. 51 for notation.

of rows (or *columns*, or *files*) to be that diagonal plane of the cube which contains the row (or column, or file) through a_{11}. And let us call the diagonal plane of files the *principal diagonal plane*.

If all the elements of any cubic determinant vanish, except those in the principal diagonal plane, the value of the determinant is equal to the value of the Scott's function formed of the elements in this plane, as they stand. (Scott, 1881.) Proof by development by files by Sec. 2.

Likewise, if all the elements are zero except those in the diagonal plane of columns (or rows) the determinant is equal to the ordinary determinant formed of the array of elements in that plane, as they stand. Proof by development by columns (or rows).

(9). If we denote as a *skew symmetric* cubic determinant, a determinant whose elements are such that all those in the principal diagonal plane vanish, while those on one side of it are got from those on the other by reflecting them in the principal diagonal plane and then changing their signs, a number of theorems for ordinary skew symmetric determinants may be generalized. The following will be typical:

A skew symmetric cubic determinant of odd order, vanishes. For, multiply each sheet through by -1. Then since n is odd, the determinant will change sign, by (4). But the effect of this operation is merely to interchange the sections and layers, and hence the value is unchanged, by (1). Hence that value must be zero. (Gegenbauer, 1883.)

(10). To form the derivative, with regard to a variable, of a cubic determinant of order n, whose elements are functions of that variable, write down the cubic determinant n times, differentiate every element in any one sheet (or layer, or section) in each determinant, not taking the same one twice, and the sum of the determinants so formed will be the required result. This can be proved at once by expanding by columns (or rows), and then differentiating by the ordinary rule.

(11). Let us now define the *minor* of any element of a cubic determinant to be the cubic determinant formed by striking out of the given determinant the three planes through this element. The *co-factor* of any element is the minor of that element taken with the sign plus or minus according as the sum of the two subscripts of the given element is even or odd. It is seen that the sign of the co-factor will be the same all along any file; and that the sign of the cofactor for the elements in any sheet is found precisely as if the array in that sheet were an ordinary determinant.

Any cubic determinant is equal to the sum of the products of the ele-

ments in any one plane multiplied by their respective co-factors. The proof is analogous to that for ordinary determinants, and depends upon the expansion of the given determinant into ordinary products. (Dahlander.) A proof by Sec. 2 is also easy.

(12). Hence, if two cubic determinants are identical except for the elements of some one given plane in each, their sum is the cubic determinant identical with either except for this same plane, any element of which is the sum of the corresponding elements in the two given determinants. (Dahlander.)

Conversely, if the elements of any plane of a cubic determinant are all polynomial, it may be written as the sum of several cubic determinants, in an obvious manner. (Dahlander.) The proof depends directly on (11).

(13). In any cubic determinant, we may add to any layer (or section; but *not* sheet) any multiple of any other layer (or section). For a determinant with two layers (or sections) one of which is a multiple of the other, vanishes, by (3) and (4). (Gegenbauer, 1881.)

(14). (a) A cubic determinant in which all of the elements in any one plane are zero except a single one, is equal to the product of that element and its co-factor. If all the elements in any plane are zero, the determinant vanishes. (Padova.) Proof by (11).

(b) Conversely, the order of any cubic determinant, by the reverse process to (a), may be raised by any number. (Padova.)

(c) A cubic determinant in which all the elements are zeros except those in the principal diagonal plane and those in some one sheet (or section, or layer), is equal to the Scott's function formed from the elements in the principal diagonal plane, as they stand. For the terms given by the Scott's functions evidently enter, by Sec. 2; and no others can enter, as is seen by forming the minor of any element in the sheet (or section, or layer) whose elements are arbitrary.

(15). Any cubic determinant of order n, is equal to the sum of the n cubic determinants formed by omitting from any one plane, in succession, all but a single line, no one being left twice and replacing the elements thus omitted by zeros. (Padova.) This is related to the theorem on expansion by minors. The proof is obvious either by (11), or by direct expansion by Sec. 2 into plane determinants (in case of rows or columns), or into Scott's functions (in case of files).

(16). *Laplace's Theorems* are easily extended to cubic determinants. Let us take any cubic determinant, and split it into two arrays: one of q sheets (or sections, or layers), and the other of $(n-q)$ sheets (or sections, or layers).

Out of the first array let us form a cubic determinant of order q by striking out of it any $(n-q)$ sections and any $(n-q)$ layers. Let us form the *conjugate determinant* from the second array, by striking out of it those q sections and those q layers *not* struck out of the first array, and prefixing to it the sign + or −, according as the term which is the product of the elements in the principal diagonals of the two determinants is positive or negative in the complete expansion of the given determinant. *The given cubic determinant is equal to the sum of the products of every such pair of conjugate determinants that can be formed.* (Gegenbauer, 1881.) The proof consists in expanding the given cubic determinant, and each of those in the result, into plane determinants by Sec. 2, and then applying Laplace's theorem to each of the resulting plane determinants of order n.

In a manner analogous to (15) we have: Any cubic determinant is equal to the sum obtained by forming all possible cubic determinants by omitting from any q parallel planes, $(n-q)$ planes of either other sort, and adding the results. (Of course, the absent elements are supplied by zeros.)

Special and generalized forms of these theorems are obvious. From one such we see that: The product of any two cubic determinants of orders p and q is expressible as a cubic determinant of order $(p+q)$, whose principal diagonal consists of the principal diagonals of the two given determinants; the position of the other elements being obvious from (16). By aid of (8), (13) and the above, the theorem of the next article may be proved in a manner analogous to that used in ordinary determinants.

4. The Multiplication Theorem.

The product of any cubic determinant of order n and any plane determinant of order n, may be expressed as a cubic determinant of order n.* In forming this product determinant, we need merely replace each sheet of the given cubic determinant by a new sheet, the elements of which are formed from the array in the old sheet, and the array of the given plane determinant, in precisely the same way in which the elements in the product of two plane determinants are formed from the arrays which compose those determinants; the same order being observed in each sheet. (Padova.) There are then four

* It can be shown that the product of a determinant of *class* p (see the second foot-note in Sec. 2) and *order* n by a determinant of *class* q and *order* n is a determinant of *class* $(p+q-2)$ and *order* n. For a *cubic* determinant, $p=3$; for a *plane* determinant $q=2$; whence $(p+q-2)$ $=3$, and the product is a *cubic* determinant.

distinct methods of multiplication, viz., those in which we combine: (1) *Columns* of the cubic determinant with *columns* of the plane determinant; (2) *Columns* (of the cubic determinant) with *rows* (of the plane determinant); (3) *Rows* with *columns*; (4) *Rows* with *rows*. We give the proof of (1). The proofs of (2), (3) and (4), are exactly like it. Expand the given cubic determinant into $n\,!$ plane determinants, by columns, by Sec. 2; multiply each of these by the given plane determinant by the ordinary method of columns and columns. The result will be $n\,!$ plane determinants of order n, in which there stand identical columns wherever there were such in the original expansion of the given cubic determinant. Hence this sum is itself a cubic determinant of order n, in which it is seen at once that the new elements are formed as stated in the above theorem.

There is no similar theorem in which the word *file* replaces the word *row* (or *column*), for there is no development by files in Sec. 2 into ordinary determinants.

It is now possible to multiply any cubic determinant by any plane determinant; for their orders may be made equal by (14), or by the similar rule in plane determinants.

5. Invariants and Covariants.

(a) Let us consider n forms, $f_1, f_2, \; . \; . \; . \; , f_n$, each of n variables $x_1, x_2, \ldots, x_n$; where the degree of each form is arbitrary. Let us *define* as the *Hessian* of these forms*, that cubic determinant of which the i th sheet is:

$$\left|\begin{array}{cccc} \frac{\partial^2 f_i}{\partial x_1 \partial x_1}, & \frac{\partial^2 f_i}{\partial x_1 \partial x_2}, & \cdots\cdots, & \frac{\partial^2 f_i}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f_i}{\partial x_2 \partial x_1}, & \frac{\partial^2 f_i}{\partial x_2 \partial x_2}, & \cdots\cdots, & \frac{\partial^2 f_i}{\partial x_2 \partial x_n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \frac{\partial^2 f_i}{\partial x_n \partial x_1}, & \frac{\partial^2 f_i}{\partial x_n \partial x_2}, & \cdots\cdots, & \frac{f \partial^2{}_i}{\partial x_n \partial x_n} \end{array}\right| (i = 1, 2, \ldots, n). \qquad (1)$$

Then the Hessian of any such set of n forms in n variables, is a simul-

* This nomenclature will cause no ambiguity. For when the number of forms is unity, the function reduces to a constant multiplied by the ordinary Hessian.

Out of the first array let us form a cubic determinant of order q by striking out of it any $(n-q)$ sections and any $(n-q)$ layers. Let us form the *conjugate determinant* from the second array, by striking out of it those q sections and those q layers *not* struck out of the first array, and prefixing to it the sign + or −, according as the term which is the product of the elements in the principal diagonals of the two determinants is positive or negative in the complete expansion of the given determinant. *The given cubic determinant is equal to the sum of the products of every such pair of conjugate determinants that can be formed.* (Gegenbauer, 1881.) The proof consists in expanding the given cubic determinant, and each of those in the result, into plane determinants by Sec. 2, and then applying Laplace's theorem to each of the resulting plane determinants of order n.

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$$\boxed{\begin{array}{cccc} \dfrac{\partial^2 f_i}{\partial x_1 \partial x_1}, & \dfrac{\partial^2 f_i}{\partial x_1 \partial x_2}, & \ldots\ldots, & \dfrac{\partial^2 f_i}{\partial x_1 \partial x_n} \\ \dfrac{\partial^2 f_i}{\partial x_2 \partial x_1}, & \dfrac{\partial^2 f_i}{\partial x_2 \partial x_2}, & \ldots\ldots, & \dfrac{\partial^2 f_i}{\partial x_2 \partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial^2 f_i}{\partial x_n \partial x_1}, & \dfrac{\partial^2 f_i}{\partial x_n \partial x_2}, & \ldots\ldots, & \dfrac{f \partial^2{}_i}{\partial x_n \partial x_n} \end{array}} (i = 1, 2, \ldots, n). \qquad (1)$$

Then the Hessian of any such set of n forms in n variables, is a simul-

* This nomenclature will cause no ambiguity. For when the number of forms is unity, the function reduces to a constant multiplied by the ordinary Hessian.

*taneous covariant of those forms.** (Gegenbauer, 1881.) To show this, let the Hessian be denoted by H. Let us perform upon the variables a linear transformation:

$$y_j = l_{1j}\,x_1 + l_{2j}\,x_2 + \cdot \cdot \cdot + l_{nj}\,x_n \qquad (j = 1, 2, \cdot \cdot \cdot \cdot, n) \tag{2}$$

whose determinant

$$D = \begin{vmatrix} l_{11}, l_{21}, & \cdot\cdot\cdot\cdot\cdot, & l_{n1} \\ l_{12}, l_{22}, & \cdot\cdot\cdot\cdot\cdot, & l_{n2} \\ \cdot\cdot\cdot\cdot & \cdot\cdot\cdot\cdot & \cdot\cdot \\ \cdot\cdot\cdot\cdot & \cdot\cdot\cdot\cdot & \cdot\cdot \\ l_{1n}, l_{2n}, & \cdot\cdot\cdot\cdot\cdot, & l_{nn} \end{vmatrix} \tag{3}$$

does not vanish. Having introduced these variables into each of the given forms, let us form the new Hessian, H', from the transformed forms, now regarding the y's as variable, in place of the x's. We will show that

$$H \cdot D^2 \equiv H'. \tag{4}$$

By (2) we see that

$$\frac{\partial}{\partial y_j} = l_{1j}\frac{\partial}{\partial x_1} + l_{2j}\frac{\partial}{\partial x_2} + \cdot \cdot \cdot \cdot + l_{nj}\frac{\partial}{\partial x_n} \quad (j = 1, 2, \cdot \cdot \cdot, n). \tag{5}$$

Now by Sec. 4, the product HD is a cubic determinant whose ith sheet is

$$\begin{vmatrix} l_{11}\dfrac{\partial^2 f_i}{\partial x_1 \partial x_1} + l_{21}\dfrac{\partial^2 f_i}{\partial x_1 \partial x_2} + \cdots + l_{n1}\dfrac{\partial^2 f_i}{\partial x_1 \partial x_n}\,; & l_{12}\dfrac{\partial^2 f_i}{\partial x_1 \partial x_1} + \cdots; & \cdots\cdot; & l_{1n}\dfrac{\partial^2 f_i}{\partial x_1 \partial x_1} + \cdots \\ l_{11}\dfrac{\partial^2 f_i}{\partial x_2 \partial x_1} + l_{21}\dfrac{\partial^2 f_i}{\partial x_2 \partial x_2} + \cdots + l_{n1}\dfrac{\partial^2 f_i}{\partial x_2 \partial x_n}\,; & l_{12}\dfrac{\partial^2 f_i}{\partial x_2 \partial x_1} + \cdots; & \cdots\cdot; & l_{1n}\dfrac{\partial^2 f_i}{\partial x_2 \partial x_1} + \cdots \\ \cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot & \cdot\cdot\cdot\cdot & \cdot\cdot & \cdot\cdot\cdot\cdot \\ \cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot & \cdot\cdot\cdot\cdot & \cdot\cdot & \cdot\cdot\cdot\cdot \\ l_{11}\dfrac{\partial^2 f_i}{\partial x_n \partial x_1} + l_{21}\dfrac{\partial^2 f_i}{\partial x_n \partial x_2} + \cdots + l_{n1}\dfrac{\partial^2 f_i}{\partial x_n \partial x_n}\,; & l_{12}\dfrac{\partial^2 f_i}{\partial x_n \partial x_1} + \cdots; & \cdots\cdot; & l_{1n}\dfrac{\partial^2 f_i}{\partial x_n \partial x_1} + \cdots \end{vmatrix}$$

$$(i = 1, 2, \cdot \cdot \cdot, n)$$

the multiplication being performed by *rows* and *rows*. This may be written by (5) in the form:

* The generalization of this and of the following theorems to m dimensions, is obvious and important. It is hoped that the attempt at simplicity in this paper will justify leaving these theorems to be inferred by analogy.

$$\left|\begin{array}{llll} \dfrac{\partial}{\partial y_1}\left(\dfrac{\partial f_i}{\partial x_1}\right); & \dfrac{\partial}{\partial y_2}\left(\dfrac{\partial f_i}{\partial x_1}\right); & \cdots\cdots; & \dfrac{\partial}{\partial y_n}\left(\dfrac{\partial f_i}{\partial x_1}\right) \\ \dfrac{\partial}{\partial y_1}\left(\dfrac{\partial f_i}{\partial x_2}\right); & \dfrac{\partial}{\partial y_2}\left(\dfrac{\partial f_i}{\partial x_2}\right); & \cdots\cdots; & \dfrac{\partial}{\partial y_n}\left(\dfrac{\partial f_i}{\partial x_2}\right) \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial}{\partial y_1}\left(\dfrac{\partial f_i}{\partial x_n}\right); & \dfrac{\partial}{\partial y_2}\left(\dfrac{\partial f_i}{\partial x_n}\right); & \cdots\cdots; & \dfrac{\partial}{\partial y_n}\left(\dfrac{\partial f_i}{\partial x_n}\right) \end{array}\right| \qquad (i=1,\,2,\,\ldots,\,n).$$

Now let us form $H\cdot D^2$, by multiplying the determinant just found by D, by Sec. 4, the multiplication being performed this time by *columns* and *rows*. By a reduction precisely similar to the above, the ith sheet becomes, by virtue of (5):

$$\left|\begin{array}{lllll} \dfrac{\partial^2 f_i}{\partial y_1\partial y_1}; & \dfrac{\partial^2 f_i}{\partial y_1\partial y_2}; & \dfrac{\partial^2 f_i}{\partial y_1\partial y_3}; & \cdots\cdots; & \dfrac{\partial^2 f_i}{\partial y_1\partial y_n} \\ \dfrac{\partial^2 f_i}{\partial y_2\partial y_1}; & \dfrac{\partial^2 f_i}{\partial y_2\partial y_2}; & \dfrac{\partial^2 f_i}{\partial y_2\partial y_3}; & \cdots\cdots; & \dfrac{\partial^2 f_i}{\partial y_2\partial y_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial^2 f_i}{\partial y_n\partial y_1}; & \dfrac{\partial^2 f_i}{\partial y_n\partial y_2}; & \cdots & \cdots\cdots; & \dfrac{\partial^2 f_i}{\partial y_n\partial y_n} \end{array}\right| \qquad (i=1,\,2,\,\ldots,\,n).$$

But such a determinant is precisely H'. Hence (4) holds. Q. E. D.

Since by (3), Sec. 3, any two sheets (or any number of them) may be identical without causing the determinant to vanish, the functions $f_1, f_2, \ldots, f_n$ *need not be distinct*, but may have some duplicates among them.* In particular, *if all n forms coincide* their Hessian reduces to $n!$ times the ordinary Hessian of that form, by (5), Sec. 3.

If all the forms are quadratic forms, the form f_i may be written:

---

* When there are fewer than n forms, several simultaneous covariants of the forms can thus be written. It is also to be remembered that a covariant of a covariant is a covariant.

$$\begin{aligned} f_i \equiv \frac{1}{2} \Big\{ & h^{(i)}_{11}x_1^2 + h^{(i)}_{12}x_1x_2 + h^{(i)}_{13}x_1x_3 + \cdot\cdot\cdot\cdot + h^{(i)}_{1n}x_1x_n \\ & + h^{(i)}_{21}x_2x_1 + h^{(i)}_{22}x_2^2 + \cdots\cdots + h^{(i)}_{2n}x_2x_n \\ & + \cdots\cdots\cdots\cdots \\ & + h^{(i)}_{n1}x_nx_1 + h^{(i)}_{n2}x_nx_2 + \cdots\cdots + h^{(i)}_{nn}x_n^2 \Big\}, \end{aligned}$$

where $h_{jk} = h_{kj}$.

The determinant H then has as its ith sheet the matrix of the coefficients $h^{(i)}$ in the form f_i. Hence the Hessian of n forms, each quadratic in n variables, is the cubic determinant formed of the matrix of the coefficients of the forms as they stand; and this is a simultaneous *invariant* of these given forms (Zehfuss.) As in the general case the forms need not be distinct, and when there are fewer than n *distinct* forms, there are several simultaneous invariants which can thus be formed.

A single illustration will be given.* Let

$$\begin{aligned} f \equiv\ & a_{11}x_1^2 + a_{12}x_1x_2 + \cdot\cdot\cdot + a_{14}x_1x_4 \\ & + a_{21}x_2x_1 + \cdots\cdots + a_{24}x_2x_4 \\ & + \cdots\cdots\cdots\cdots \\ & + a_{41}x_4x_1 + \cdots\cdots + a_{44}x_4^2 = 0 \end{aligned}$$

be the equation of a quadric surface; and let

$$\phi \equiv lx_1 + mx_2 + nx_3 + px_4 = 0$$

be the equation of a plane. Then let us take f, f, f, and ϕ^2 as our four ($n = 4$) quadratic forms. It follows that:

$$\begin{array}{|l|l|l|llll|} \hline a_{11}, & a_{12}, & a_{13}, & a_{14} & & & \\ a_{21}, & a_{11}, & a_{12}, & a_{13}, & a_{14} & & \\ a_{31}, & a_{21}, & a_{11}, & a_{12}, & a_{13}, & a_{14} & \\ a_{41}, & a_{31}, & a_{21}, & l^2, & lm, & ln, & lp \\ & x_{41}, & a_{31}, & ml, & m^2, & mn, & mp \\ & & a_{41}, & nl, & nm, & n^2, & np \\ & & & pl, & pm, & pn, & p^2 \\ \hline \end{array} \equiv H, \text{ (say)},$$

---

* The similar condition for a line tangent to a conic is readily obtainable.

is an invariant of f and ϕ. But, by a suitably chosen linear transformation, we can reduce f to a sum of squares: $f' \equiv a'_{11}x'^2_1 + a'_{22}x'^2_2 + a'_{33}x'^2_3 + a'_{44}x'^2_4 = 0$, and ϕ goes into $\phi' \equiv l'x'_1 + m'x'_2 + n'x'_3 + p'x'_4 = 0$. Whence H' is a cubic determinant in which the elements in the first three sheets are zeros, except the diagonals which are each $a'_{11}, a'_{22}, a'_{33}, a'_{44}$. Hence by (14) c, Sec. 3,

$$H' = 6 \left\{ l'^2 a'_{22}a'_{33}a'_{44} + m'^2 a'_{11}a'_{33}a'_{44} + n'^2 a'_{11}a'_{22}a'_{44} + p' a'_{11}a'_{22}a'_{33} \right\}.$$

But the vanishing of this expression is easily shown to be the necessary and sufficient condition that the plane $\phi' = 0$ be tangent to the surface $f' = 0$. Since H is invariant, it follows that $H = 0$ *is the necessary and sufficient condition that the given plane* $\phi = 0$ *be tangent to the given quadric* $f = 0$. This form of the condition is sometimes more convenient and symmetrical than that ordinarily given, to which it is of course equivalent. If l, m, n, p be regarded as the *plane* (or "*tangential*") *coordinates* of any plane in space, $H = 0$ is the equation of the quadric surface in plane coordinates. It is seen at once to be of the second degree.

The process here used is applicable generally; *i. e.*, in place of any of the forms f_i, we may put any power of itself, or more generally still, any form which is a function of all the given forms. The Hessian will still be a simultaneous covariant of the given set of forms.

(b) Another general set of covariants can be produced as follows: Let $f_1, f_2, \ldots, f_n$ be a set of n *distinct* forms in n variables $x_1, x_2, \ldots, x_n$. Let us form the cubic determinant whose ith sheet is:

$$\left| \begin{array}{cccc} \frac{\partial f_i}{\partial x_1}\frac{\partial f_i}{\partial x_1}; & \frac{\partial f_i}{\partial x_1}\frac{\partial f_i}{\partial x_2}; & \ldots\ldots; & \frac{\partial f_i}{\partial x_1}\frac{\partial f_i}{\partial x_n} \\ \frac{\partial f_i}{\partial x_2}\frac{\partial f_i}{\partial x_1}; & \frac{\partial f_i}{\partial x_2}\frac{\partial f_1}{\partial x_2}; & \ldots\ldots; & \frac{\partial f_i}{\partial x_2}\frac{\partial f_i}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f_i}{\partial x_n}\frac{\partial f_i}{\partial x_1}; & \frac{\partial f_i}{\partial x_n}\frac{\partial f_i}{\partial x_2}; & \ldots\ldots; & \frac{\partial f_i}{\partial x_n}\frac{\partial f_i}{\partial x_n} \end{array} \right| (i = 1, 2, \ldots, n).$$

The cubic determinant thus formed will be a simultaneous covariant of the given forms. (von Escherich.) The proof is practically a repetition of that

given in (a), above, except that we have a product in place of a second derivative.

In case all of the forms f_i are linear, this invariant is evidently the same as the Hessian of the squares of the forms, except for the constant factor 2^n.

(c) It is at once apparent that we may combine (a) and (b). Given any set of forms $f_1, f_2, \ldots, f_n$, in n variables $x_1, x_2, \ldots, x_n$, let us form the cubic determinant by performing on any q of the forms $(0 \leqq q \leqq n)$ *the Hessian operation* (as I shall call the operation in (a)), to get q of the sheets of the cubic determinant; and then performing on the remaining $(n-q)$ forms *the Product operation* (as I shall call the operation in (b)), to get the other $(n-q)$ sheets of the cubic determinant.

The cubic determinant thus formed will be a simultaneous covariant of the given forms. The proof consists in a repetition of the proofs given for (a) and (b).

It is to be noted that the forms upon which the *Product operation* is performed must be *distinct* from each other, here and in (b), as else the cubic determinant vanishes by (7), Sec. 3.

The particular example under (a) can be regarded as formed by the Hessian operation on f, three times, followed by the Product operation on ϕ.

(d) Next, let any two sets of forms $f_1^{(1)}, f_2^{(1)}, \ldots, f_n^{(1)}$ and $f_1^{(2)}, f_2^{(2)}, \ldots, f_n^{(2)}$ be given, and let us form the cubic determinant whose ith sheet is

$$\left|\begin{array}{cccc} \dfrac{\partial f_i^{(1)}}{\partial x_1}\dfrac{\partial f_i^{(2)}}{\partial x_1}; & \dfrac{\partial f_i^{(1)}}{\partial x_1}\dfrac{\partial f_i^{(2)}}{\partial x_2}; & \cdots\cdots\cdots\cdots; & \dfrac{\partial f_i^{(1)}}{\partial x_1}\dfrac{\partial f_i^{(2)}}{\partial x_n} \\ \dfrac{\partial f_i^{(1)}}{\partial x_2}\dfrac{\partial f_i^{(2)}}{\partial x_1}; & \dfrac{\partial f_i^{(1)}}{\partial x_2}\dfrac{\partial f_i^{(2)}}{\partial x_2}; & \cdots\cdots\cdots\cdots; & \dfrac{\partial f_i^{(1)}}{\partial x_2}\dfrac{\partial f_i^{(2)}}{\partial x_n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \dfrac{\partial f_i^{(1)}}{\partial x_n}\dfrac{\partial f_i^{(2)}}{\partial x_1}; & \dfrac{\partial f_i^{(1)}}{\partial x_n}\dfrac{\partial f_i^{(2)}}{\partial x_2}; & \cdots\cdots\cdots\cdots; & \dfrac{\partial f_i^{(1)}}{\partial x_n}\dfrac{\partial f_i^{(2)}}{\partial x_n} \end{array}\right| \quad (i=1, 2, 3, \cdots, n).$$

The cubic determinant thus formed will be a simultaneous covariant of the given forms. (von Escherich.) The proof is precisely as in (a), (b), or (c), except that we have to take into account now, that the layers and sections may be interchanged, by 1, Sec. 3. As in (b) and (c), we must avoid having two forms of the same set equal. But any number of forms of one set may

be equal to forms in the other set. I will call this operation, the *Double Product operation.*

(*e*) As in (*c*) we may combine the operations of (*a*), (*b*), (*d*). Thus having given any set of forms $f_1, f_2, \ldots$ in n variables $x_1, x_2, \ldots, x_n$ we may form a cubic determinant of which q sheets $(0 \leqq q \leqq n)$ are formed by the Hessian operation performed on any q of the given forms; p sheets $(0 \leqq p \leqq n-q)$ by the Product operation performed on any p of the given forms; and the remaining r $(= n-p-q)$ sheets, by the Double Product operation performed on any two sets of r forms each, taken from the given forms; *and the cubic determinant thus formed will be a simultaneous covariant of the forms used.** The proof is the same as before. It is to be noticed that for particular values of p and q, this theorem covers all the previous cases, (*a*), (*b*), (*c*), (*d*). In forming the determinant, some of the forms may be identical and we need only be careful to make any of them distinct in case we desire to guard against the identical vanishing of the determinant, as in (*b*), (*c*), and (*d*).

6. Analogy to Ordinary Products.

(1). *Notation.* Let us denote by $[a\ b\ c\ \ldots\ q]$ a cubic determinant of order n, in which the first sheet is:

$$\begin{vmatrix} a_{11}, & a_{12}, & a_{13}, & \ldots\ldots, & a_{1n} \\ a_{21}, & a_{22}, & \ldots\ldots & \ldots\ldots, & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1}, & a_{n2}, & \ldots\ldots & \ldots\ldots, & a_{nn} \end{vmatrix}$$

the successive sheets merely changing the *dominant letter*. There are always n dominant letters, and of each of these there are n^2 with different subscripts.

(2). *Some simple analogies.* In ordinary algebra we have: $ab \equiv ba$. Likewise here: $[ab] \equiv [ba]$ by (2), Sec. 3,

$$i.\ e. \quad \begin{array}{|ll|}\hline a_{11}, & a_{12} \\ a_{21}, & \begin{array}{|ll|}\hline b_{11}, & b_{12} \\ b_{21}, & b_{22} \\ \hline\end{array} \\ \hline\end{array} \equiv \begin{array}{|ll|}\hline b_{11}, & b_{12} \\ b_{21}, & \begin{array}{|ll|}\hline a_{11}, & a_{12} \\ a_{21}, & a_{22} \\ \hline\end{array} \\ \hline\end{array}.$$

* The generalization of this theorem to m dimensions gives a very general theorem. It is believed that this is obvious without the rather complex statement of it.

In ordinary algebra: $(a+b)c \equiv ac + bc.$

Likewise here: $[(a+b)c] \equiv [ac] + [bc]$ by (12), Sec. 3,

$$i.\,e.\quad \boxed{\begin{array}{l} a_{11}+b_{11},\ a_{12}+b_{12} \\ a_{21}+b_{21},\ \boxed{\begin{array}{ll} c_{11}, & c_{12} \\ c_{21}, & c_{22} \end{array}} \end{array}} \equiv \boxed{\begin{array}{l} a_{11},\ a_{12} \\ a_{21},\ \boxed{\begin{array}{ll} c_{11}, & c_{12} \\ c_{21}, & c_{22} \end{array}} \end{array}} + \boxed{\begin{array}{l} b_{11},\ b_{12} \\ b_{21},\ \boxed{\begin{array}{ll} c_{11}, & c_{12} \\ c_{21}, & c_{22} \end{array}} \end{array}}.$$

Laplace's theorem (see (16), Sec. 3) may evidently be stated for $n = 4$ as follows:

$$[abcd] \equiv \sum [ab]\cdot[cd]$$

where Σ indicates the sum of all possible combinations such as that indicated. In the use of such forms, however, considerable care is necessary, and they are avoided in what follows.

Any simple homogeneous algebraic formula is seen at once to have an analogue in cubic determinants.

Take $a^2 + 2ab + b^2 \equiv (a+b)^2.$

We have $[aa] + 2\,[ab] + [bb] \equiv [(a+b)(a+b)],$

$$i.e.\quad \boxed{\begin{array}{l} a_{11},\ a_{12} \\ a_{21},\ \boxed{\begin{array}{ll} a_{11}, & a_{12} \\ a_{21}, & a_{22} \end{array}} \end{array}} + 2\,\boxed{\begin{array}{l} a_{11},\ a_{12} \\ a_{21},\ \boxed{\begin{array}{ll} b_{11}, & b_{12} \\ b_{21}, & b_{22} \end{array}} \end{array}} + \boxed{\begin{array}{l} b_{11},\ b_{12} \\ b_{21},\ \boxed{\begin{array}{ll} b_{11}, & b_{12} \\ b_{21}, & b_{22} \end{array}} \end{array}} \equiv \boxed{\begin{array}{l} a_{11}+b_{11},\ a_{12}+b_{12} \\ a_{21}+b_{21},\ \boxed{\begin{array}{l} a_{11}+b_{11},\ a_{12}+b_{12} \\ a_{21}+b_{21},\ a_{22}+b_{22} \end{array}} \end{array}}.$$

It is seen by (5), Sec. 3, that any square term gives a cubic determinant which degenerates into a plane determinant. Thus the above becomes:

$$\begin{vmatrix} a_{11}, & a_{12} \\ a_{21}, & a_{22} \end{vmatrix} + \boxed{\begin{array}{l} a_{11},\ a_{12} \\ a_{21},\ \boxed{\begin{array}{ll} b_{11}, & b_{12} \\ b_{21}, & b_{22} \end{array}} \end{array}} + \begin{vmatrix} b_{11}, & b_{12} \\ b_{21}, & b_{22} \end{vmatrix} \equiv \begin{vmatrix} a_{11}+b_{11}, & a_{12}+b_{12} \\ a_{21}+b_{21}, & a_{22}+b_{22} \end{vmatrix}.$$

Likewise $[aa] - [bb] \equiv [(a-b)(a+b)]$. And so on.

(3) In fact we may state at once *a Law of Analogy*:

To any homogeneous formula in ordinary algebra there corresponds a

theorem in cubic determinants. This follows from the fact that we may start with the formulæ

$$[ab] = [ba]$$

and

$$[(a+b)c] = [ac] + [bc]$$

and build up the cubic determinant theorem by methods exactly analogous to those used in ordinary algebra.

(4) *Examples.* Several examples follow, in each of which a direct verification is easy.

$$[\,(x+y+z)^3] - [x^3] - [y^3] - [z^3] \equiv 3\,[\,(y+z)\,(z+x)\,(x+y)\,].$$

Since each cube gives a degenerating cubic determinant, we get, by (5), Sec. 3

$$2\begin{vmatrix} x_{11}+y_{11}+z_{11}\,; x_{12}+y_{12}+z_{12}\,; x_{13}+y_{13}+z_{13} \\ x_{21}+y_{21}+z_{21}\,; x_{22}+y_{22}+z_{22}\,; x_{23}+y_{23}+z_{23} \\ x_{31}+y_{31}+z_{31}\,; x_{32}+y_{32}+z_{32}\,; x_{33}+y_{33}+z_{33} \end{vmatrix} - 2\begin{vmatrix} x_{11}\,; x_{12}\,; x_{13} \\ x_{21}\,; x_{22}\,; x_{23} \\ x_{31}\,; x_{32}\,; x_{33} \end{vmatrix} - 2\begin{vmatrix} y_{11}\,; y_{12}\,; y_{13} \\ y_{21}\,; y_{22}\,; y_{23} \\ y_{31}\,; y_{32}\,; y_{33} \end{vmatrix}$$

$$-2\begin{vmatrix} z_{11}\,; z_{12}\,; z_{13} \\ z_{21}\,; z_{22}\,; z_{23} \\ z_{31}\,; z_{32}\,; z_{33} \end{vmatrix} \equiv \begin{array}{|l} y_{11}+z_{11}\,;\; y_{12}+z_{12}\,;\; y_{13}+z_{13} \\ y_{21}+z_{21}\,;\; \begin{array}{|l} z_{11}+x_{11}\,; z_{12}+x_{12}\,; z_{13}+x_{13} \\ z_{21}+x_{21}\,;\; \begin{array}{|l|} x_{11}+y_{11}\,; x_{12}+y_{12}\,; x_{13}+y_{13} \\ x_{21}+y_{21}\,; x_{22}+y_{22}\,; x_{23}+y_{23} \\ x_{31}+y_{31}\,; x_{32}+y_{32}\,; x_{33}+y_{33} \end{array} \\ z_{31}+x_{31}\,; \end{array} \\ y_{31}+z_{31}\,; \end{array}\;;$$

a rather interesting result, especially for some special values of the $(x,\ y,\ z)$'s, and it is easily verified.

The binomial theorem is interesting. We find:

$$[(a+b)^n] \equiv [a^n] + n\,[a^{n-1}b] + \frac{n(n-1)}{2\,!}\,[a^{n-2}b^2] + \cdots\cdot$$

$$+\frac{n(n-1)\cdot\cdots\cdot(n-r+1)}{r\,!}\,[a^{n-r}b^r] + \cdots\cdot$$

$$+\frac{n(n-1)}{2\,!}\,[a^2b^{n-2}] + n\,[ab^{n-1}] + [b^n],$$

where n is of course integral. Whence by (5) and (6), Sec. 3, if we denote by $S^{(j)}_{(n-j)}$ the sum of the $\dfrac{n\,!}{(n-j)\,!\,j\,!}$ plane determinants which can be formed by substituting any j rows of the plane determinant of the matrix of the b's, for the corresponding j rows of the plane determinant of the matrix of the a's; and if we denote by D the determinant of the matrix of the $(a+b)$'s, then:

$$n!D \equiv n!\,(S^{(0)}_{(n)}) + n\Big((n-1)!\cdot 1!\,S^{(1)}_{(n-1)}\Big) + \frac{n(n-1)}{2!}\Big((n-2)!\,2!\,S^{(2)}_{(n-2)}\Big) + \cdots$$
$$\cdots\cdot + \frac{n(n-1)\cdots(n-r+1)}{r!}\Big((n-r)!\,r!\,S^{(r)}_{(n-r)}\Big) + \cdots + n!(S^{(n)}_{(0)}),$$

which reduces at once to

$$D \equiv S^{(0)}_{(n)} + S^{(1)}_{(n-1)} + \cdots\cdot + S^{(r)}_{(n-r)} + \cdots\cdot + S^{(n-1)}_{(1)} + S^{(n)}_{(0)};$$

a very peculiar result, which can be verified immediately.

Similarly:

$$\Big[\prod_{i=1}^{i=n}(a^{(i)} + b^{(i)})\Big] = \Big[\prod_{i=1}^{i=n}(a^{(i)})\Big] + \sum_{j=1}^{j=n}\Big[\prod_{j\neq i=1}^{j\neq i=n-1}(a^{(i)}b^{(j)})\Big] + \cdots$$
$$\cdots + \Sigma\Big[\prod_{i=1}^{i=n-r} a^{(i)} \prod_{j=n-r}^{j=n} b^{(j)}\Big] + \cdots\cdot + \Big[\prod_{i=1}^{i=n}(b^{(i)})\Big],$$

where, as usual, Π denotes a product and Σ a summation. This of course includes the last example. It also includes the case where $a^{(i)} = a^{(1)}$ for all values of i. This gives:

$$\Big[\prod_{i=1}^{i=n}(a + b^{(i)})\Big] = \Big[a^n\Big] + \sum_{j=1}^{j=n}\Big[a^{n-1}b^{(j)}\Big] + \cdots + \Sigma\Big[a^{n-r}\prod_{j=1}^{j=r}(b^{(j)})\Big]$$
$$+ \cdots\cdot + \Sigma\Big[a\prod_{j=1}^{j=n-1}(b^{(j)})\Big] + \Big[\prod_{i=1}^{i=n}(b^{(i)})\Big].$$

Let us now specialize the b's.* Let the cubical determinant $D \equiv \Big[\prod_{i=1}^{i=n}(a + b^{(i)})\Big]$ have each plane identical with the matrix of a's, except that in the ith sheet, the ith column (or row) is multiplied by $(1+h)$. Then the cubic determinant $\Big[\prod_{i=1}^{i=n}(b^{(i)})\Big]$ will have all its elements equal to zero, except that in the ith sheet, the ith row is h times the corresponding row in $[a^n] \equiv D'$ (say). We find at once by (5) and (6), Sec. 3:

$$D \equiv D'\Big\{1 + \frac{1}{(n-1)!\,1!}(n-1)!\,h + \cdots + \frac{1}{(n-r)!\,r!}(n-r!)h^r + \cdots + \frac{1}{n!}h^n\Big\}$$
$$\equiv D'\Big\{1 + \frac{h}{1!} + \frac{h^2}{2!} + \cdots + \frac{h^r}{r!} + \cdots + \frac{h^n}{n!}\Big\}.$$

* I owe this particular example to F. R. Scott (*Proc. Lond. Math. Soc.*, XIII), whose proof, of course, differs from mine.

Whence

$$\underset{n=\infty}{Lim}\left(\frac{D}{D'}\right)=e^{h};$$

another peculiar result.

7. Extension to m Dimensions.

It is to be noticed that in extending these theorems to a determinant of class m (*i. e.* in m dimensions) no difficulty is experienced since our arrays do not really depend upon the geometrical form which we have so far given them. But the use of such determinants becomes more complicated as the possibility of geometrical representation of the matrix is lost.

The theorems themselves are easily suggested by those here given, and by the ordinary theorems in plane determinants. The proofs are usually easy, either directly, in the case of any particular dimension, or by a process of mathematical induction, in general, the determinant of class m being defined, as in the second foot-note in Sec. 2, as the sum of $n!$ determinants of class $(m-1)$.

It is noticed immediately that any determinant whose class is even is symmetrical with respect to all $2p$ dimensions. But a determinant of odd class (*e. g.* a cubic determinant) is unsymmetrical with respect to one dimension, but symmetrical with respect to the remaining $2p$ dimensions.

A determinant whose class is unity is evidently an ordinary product. It is interesting to notice that all general rules for m dimensional determinants apply to products when $m=1$. This in fact accounts for the analogy between cubic determinants and ordinary products, mentioned in Sec. 6, and formed the basis of a great part of the work in this paper.

We see at once that this same relation of analogy holds for any odd-dimensional determinant, for similar reasons. It is, perhaps, less interesting to notice that a similar relation of analogy holds between plane determinants (or in fact any determinants of even class), and the algebra of "alternate numbers," these alternate numbers having been defined, by Clifford, exactly so as to satisfy the laws of ordinary determinants.*

Cambridge, Mass., April, 1899.

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* Compare this fact with the use made of alternate numbers by F. R. Scott, *Theory of Determinants.*

ON TIDE CURRENTS IN ESTUARIES AND RIVERS.*

By Ernest W. Brown.

In popular and elementary text-books where the phenomena exhibited by the tides are described, a statement is frequently made which almost invariably causes difficulty because it appears to contradict every-day experience. This difficulty arises partly from insufficient explanation and partly from a want of completeness in giving the conditions which must hold in order that it may be true. In many cases of wave motion it is easier to give explanation by means of a diagram showing the form of the wave rather than the curves described by the individual particles which partake of the wave motion. In certain cases of tide waves, however, I shall try to show how the explanations can be considerably simplified by considering the motion of a single drop of the water or, let us say, of a float on the surface.

The particular statement referred to above is that in rivers, slack water (that is, no current) at any place occurs when the water is at mean level, while the maximum flow (upward current) occurs at high water and maximum ebb (downward current) at low water. This, of course, is a purely theoretical result and is quite contrary to what is usually noticed, namely, that slack water occurs very near high and low water. It demands that the bed of the river shall not be uncovered at any time, that its breadth and sectional area shall be uniform, that friction be neglected, that there be no proper current of the river and that the wave shall not change its shape as it proceeds, the last condition not holding even theoretically. Moreover, the observations must be made at places on the canal very far distant from its mouth. The place of observation is the principal cause of the difference between the theoretical and the observed results. The tidal currents are usually noticed comparatively near the mouth; if we go further up the river so as to be free from this difficulty, the friction, change of shape of the bed, etc., play important parts, and a great difference between the theoretical and observed motions still remains.

When a wave passes over the surface of still water, each particle of water describes a closed curve whose shape depends on that of the wave. With ordinary waves on deep water raised, for example, by a ship, this curve is an oval nearly symmetrical about both vertical and horizontal axes, as a little ob-

* This paper was read before the American Mathematical Society at the meeting of April 29, 1899.

servation on a float will easily show. With ordinary sea waves in deep water the vertical axis is much greater than the horizontal axis. Near the beach, that is, in shallow water, the reverse takes place, and what is noticed mainly is the motion of the float to and fro rather than up and down.

Consider a straight uniform canal with vertical sides and horizontal bed in which the water is sufficiently deep so that the waves we shall consider do not leave it dry at any time. A series of similar waves passing along the length of the canal will cause a float to describe a nearly symmetrical oval curve. If the wave length (distance from crest to crest) is not long compared with the depth of the water, the float will have but little horizontal motion. As the wave length increases, the height remaining the same, the horizontal motion will increase. When the height of the wave is a few feet, the depth of the canal a few fathoms and the wave-length several miles, a person standing on the bank notices mainly the horizontal motion. If, however, he follow the float by walking along a horizontal bank, he will notice that the float is gradually rising or falling. The rule is that the water is flowing in the direction of the wave motion when the float is above its mean level, and in the opposite direction when it is below its mean level. How this takes place will be seen at once from fig. 1.

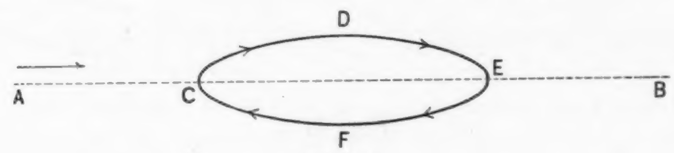

Fig. 1.

Here $A\ B$ represents the mean level of the water in the direction of the length of the canal, $C\ D\ E\ F$ the curve described by the float. When it is at D there is high water at that part of the canal, with maximum current in the direction of the arrow on the left which shows the direction of motion of the wave. At C and E we have slack water and mean level; and so on.

Suppose now the canal communicates with the sea at one end. The tidal motion in the canal is mainly caused by the tides of the ocean, that is, by a rise and fall of the ocean at the mouth of the canal. At a great distance from the mouth, the motion in our canal would still be like that which has just been described, the length $C\ E$ being some miles and $D\ F$ a few feet. If the ocean be deep and the tide wave advance in a direction perpendicular to the cliffs, the motion of a float some distance from the land will be mainly vertical. Nearer

the mouth the horizontal motion begins to be perceptible; and just inside it becomes considerable. If we have a series of floats extending from far out into the ocean right up the canal, the curves described by them vary from ovals with horizontal axes of a few feet to ovals with horizontal axes of several miles, the vertical axes being never more than a few feet. Moreover, the ovals described by floats near the mouth are no longer symmetrical, and the axes depart from the horizontal and vertical.

In fig. 2, AMB represents mean level, AM in the ocean and MB in the canal. The points D, F represent high and low water respectively in the several positions; the points E and C slack water. Thus near the mouth slack water occurs very near high and low water. This agrees with observation. Moreover, it is observed that the current "flows" some time after high water and "ebbs" some time after low water; this is illustrated in the parts DE and FC of the curves II, III, and IV the water falling and flowing from D to E and rising and ebbing from F to C; from C to D we have rising water with flowing current, and from E to F falling water with ebbing current.

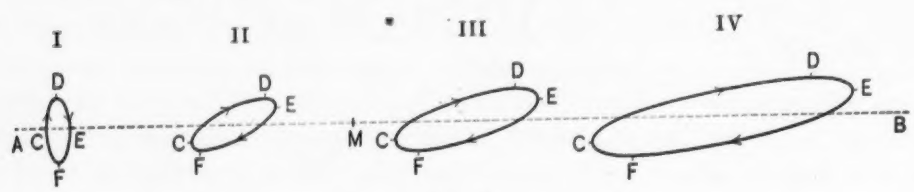

Fig. 2.

This general effect is modified by other circumstances to a very considerable extent in some rivers. Most rivers gradually decrease in depth and breadth as we get farther away from the mouth, causing (as theory indicates) an increased height of the wave and alteration of its length as it progresses up the river. In general the motion of the float at the end of the longer axis in curves III and IV is very slow compared to that at the ends of the shorter axis. In some rivers however this is not so. At C it may become very quick when it gives rise to the "bore," which takes the appearance of a bank of water, sometimes several feet in height, travelling up the river and causing the water at the places it passes to rise suddenly. The curve described has also lost all symmetry. All these phenomena are fully described in Professor Darwin's book, *The Tides*. The reader will be able to find the shape of the curves described by floats in such cases for himself.

There is no attempt at accuracy in the diagrams. These are merely intended to give a general idea of the relation of height to ebb and flow. The vertical distances are of course much exaggerated.

Haverford College, November, 1899.

NOTE ON NETTO'S THEORY OF SUBSTITUTIONS.

By G. A. Miller.

In finding a function that belongs to a given group, Netto employs a method which is apt to mislead the student. Since some other authors express themselves somewhat indefinitely* in regard to this fundamental problem, it may be desirable to give several elementary illustrations which will exhibit the points in question. We shall first show how the given method may lead to incorrect results. Employing the same Galois function ($\phi_1 = x_1 + ix_2 - x_3 - ix_4$) as Netto employs† we proceed to find a function belonging to the group $G_2 = 1$, $(x_1x_2)(x_3x_4)$ by the first method which he employs to find a function belonging to the group

$$G_8 = 1,\ (x_1x_2)(x_3x_4),\ (x_1x_3)(x_2x_4),\ (x_1x_4)(x_2x_3),\ (x_1x_3),\ (x_2x_4),\ (x_1x_2x_3x_4),\ (x_1x_4x_3x_2).$$

Transforming ϕ_1 by the substitutions of G_2 we have:

$$\psi_1 = \phi_1\phi_2 = (x_1 + ix_2 - x_3 - ix_4)(x_2 + ix_1 - x_4 - ix_3) = i(x_1 - x_3)^2 + i(x_2 - x_4)^2.$$

If we transform ϕ_1 by the substitutions of $G_2' = 1$, (x_2x_4) we obtain:

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* Vogt, *Résolution algébrique des équations*, 1895, p. 23.

† Netto, *Theory of Substitutions*, translated by Cole, 1892, p. 30.

$\psi_2 = -i\phi_1\phi_2 = (x_1 + ix_2 - x_3 - ix_4)(x_1 + ix_4 - x_3 - ix_2) = (x_1 - x_3)^2 + (x_2 - x_4)^2$.
Finally, we transform ϕ_1 by the substitutions of $G''_2 = 1, (x_1x_3)(x_2x_4)$ and obtain:
$\psi_3 = -\phi_1^2 = (x_1 + ix_2 - x_3 - ix_4)(x_3 + ix_4 - x_1 - ix_2) = -(x_1 + ix_2 - x_3 - ix_4)^2$.

It is evident that each of the functions ψ_1, ψ_2, admits all the substitutions of G_8. They admit no other substitution in these elements since G_8 is not contained in any group in these elements besides the symmetric group,* and since they clearly do not admit the transposition (x_1x_4). Hence each one of these two functions belongs to G_8 and not to the given groups of order two. This shows that the method which Netto employed to construct a function belonging to G_8 cannot be employed (if the given Galois function is used as the fundamental function) to construct a function belonging to either of the two given groups of order two. It may readily be verified that similar considerations in regard to the other two non-selfconjugate subgroups of order two that are contained in G_8 lead to the functions $-\phi_1\phi_2$ and $i\phi_1\phi_2$ respectively, each of which belongs to G_8.

With respect to G''_2 the matter is entirely different. ψ_3 actually belongs to this group since it clearly admits its substitutions and no others. These results follow very easily from the fact that the cyclical subgroup of order four contained in G_8 transforms ϕ_1 into itself multiplied by the four fourth roots of unity, while the remaining four substitutions of G_8 transform it into ψ_2 multiplied by the same four roots. Hence we observe that the given method together with the given fundamental Galois function can be employed in only one out of the five cases to construct a function belonging to a group of order two contained in G_8. From this it follows directly that this method can be employed in only one of the three cases to construct a function belonging to a group of order four that is contained in G_8, viz., when this group of order four is cyclical.†

In order to see clearly where the difficulty lies it may be well to observe that if a substitution transforms a function into itself multiplied by a constant it is necessary that the absolute value of this constant be unity; for from the equation $s^{-1}Fs = \rho F$ we obtain $s^{-n}Fs^n = \rho^n F = F$, where n is the order of s. If we transform any linear function of n independent elements by any substitution in these elements, the result is evidently equivalent to transforming its coefficients in the reverse order. Hence *s cannot transform a linear Galois function into itself multiplied by a constant* (ρ), *which is not unity, unless all*

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* Cf. Cayley, *Quarterly Journal of Mathematics*, vol. 25, 1891, p. 77.

† Vogt, *l. c.*

the coefficients of these elements can be divided into distinct sets containing a multiple of n numbers and having a common absolute value. In particular, any linear Galois function in which the absolute value of one coefficient is larger than the absolute value of any one of the others cannot be transformed (by means of a substitution) into itself multiplied by a constant.

If we employ such a Galois function (f_1) as a fundamental function, the method employed by Netto will clearly always lead to a function (F) belonging to the group: for if $f_1, f_2, \ldots, f_\lambda$ are the functions obtained by transforming f_1 by the substitutions of this group (T), then $F = f_1 f_2 f_3 \ldots f_\lambda$ is evidently transformed into itself by T. That it is not transformed into itself by any other substitution follows from the fact that such a substitution would transform the given linear factors of F into a new set none of which could be obtained by multiplying a factor of F by a constant. Since an expression can be resolved into linear factors in only one way,* the product of this second set of factors could not be equal to F. If we assume that a linear Galois function contains a constant term which is not zero, it is well known† that this also may always be employed to construct a function (according to the given method) that belongs to any given group.

CORNELL UNIVERSITY, OCTOBER, 1899.

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* Cf. Weber, *Algebra*, Vol. 1, 1898, p. 74.

† Cf. Vogt, *Résolution algébrique des équations*, 1895, p. 23.

A METHOD OF SOLVING DETERMINANTS.

By G. Macloskie.

By a simple *Method of Compression* we can reduce a determinant to another of lower order, so as greatly to lessen the labor of its evaluation. This method depends on the following theorem:

$$\begin{vmatrix} a & b & c & \cdot & \cdot \\ e & f & g & \cdot & \cdot \\ j & k & l & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} = a\ b\ c \begin{vmatrix} 1 & 1 & 1 & \cdot\ \cdot \\ \frac{e}{a} & \frac{f}{b} & \frac{g}{c} & \cdot\ \cdot \\ \frac{j}{a} & \frac{k}{b} & \frac{l}{c} & \cdot\ \cdot \\ \cdot & \cdot & \cdot & \cdot\ \cdot \end{vmatrix}$$

$$= a\ b\ c \begin{vmatrix} \left(\frac{f}{b} - \frac{e}{a}\right) & \left(\frac{g}{c} - \frac{e}{a}\right) & \cdot\ \cdot \\ \left(\frac{k}{b} - \frac{j}{a}\right) & \left(\frac{l}{c} - \frac{j}{a}\right) & \cdot\ \cdot \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot \end{vmatrix} = a \begin{vmatrix} \left(f - \frac{b}{a}e\right) & \left(g - \frac{c}{a}e\right) & \cdot\ \cdot \\ \left(k - \frac{b}{a}j\right) & \left(l - \frac{c}{a}j\right) & \cdot\ \cdot \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot \end{vmatrix}.$$

We have thus reached a new determinant of a lower order which, when multiplied by one of the elements of the original, is equal to the original determinant.

We can obtain this result directly from the original by the following general rule:

$$\begin{array}{c} \begin{vmatrix} a^* & \underline{b} & \underline{c} & \cdot & \cdot \\ \underline{e} & f & g & \cdot & \cdot \\ \underline{j} & k & l & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} \\ \hline \quad -\frac{b}{a} \quad -\frac{c}{a} \quad \cdot\ \cdot \end{array}$$

Select any of the elements as the *chief element* (as a in the margin): and mark it and all the other elements of its row and column. Form *compressors* by dividing the chief element into the other elements of its own row, writing each quotient, with the sign changed, under its proper column. Then calling the marked elements of the column to which the chief element belongs *co-compressors*, the required result is got by the following

Rule.— By taking as elements of a new determinant the remaining (unmarked) *elements of the original determinant, each increased by the product of its compressor and its co-compressor, we obtain a determinant of lower order, which, when multiplied by the chief element, is equal to the original determinant.*

NOTES. (1) If a line drawn diagonally from the original position of the chief element strikes the anterior or the upper margin at an *even* position, reckoning from the upper-anterior corner, we must prefix a minus sign to the new determinant: not so when such line strikes an *odd* marginal position.

(2) We can rectify a fractional compressor, by multiplying itself and all the members of its column by its denominator, which must then be annexed as an external divisor. (See example 1, column 3.)

(3) If there is a unit among the elements, by selecting it as chief element, we make the other elements of its row taken with the proper signs the compressors. (Example 2, stage 1.)

(4) If zeros occur as compressors or co-compressors, the elements of the columns or rows of the early stage are carried over unchanged into the new stage.

(5) Hence we learn that the common method of reducing a line of a determinant to zeros, excepting one element, and thus lowering its order, is only a particular case of the compression method (as in stages 2 and 3 of example 2).

Example 1.

$$\begin{array}{c} \begin{array}{rrrr} & & (\times 5) & \end{array} \\ \left|\begin{array}{rrrr} \underline{25} & -15 & 23 & -5 \\ -\underline{15} & -10 & 19 & 5 \\ \underline{23} & 19 & -15 & 9 \\ -5^{*} & \underline{5} & \underline{9} & -\underline{5} \end{array}\right| \\ \begin{array}{rrrr} & 1 & \frac{9}{5} & -1 \end{array} \end{array} \div 5 = \frac{5}{5} \begin{array}{c} \left|\begin{array}{rrr} 10^{*} & \underline{340} & -\underline{30} \\ -\underline{25} & -40 & 20 \\ \underline{42} & 132 & -14 \end{array}\right| \\ \begin{array}{rrr} & -34 & 3 \end{array} \end{array}$$

$$= 10 \left|\begin{array}{rr} 810 & -55 \\ -1296 & 112 \end{array}\right| = 10(90720 - 71280) = 194400.$$

Here the term selected as chief (-5 at foot of 1st column) being in an

even marginal position, appears as $+5$ when multiplied by the next determinant. In order to rectify the fractional compressor $9/5$ of the first determinant we multiply it and its column by its denominator 5 which is written above its column, and is also written as an external divisor.

The first unmarked element of the first determinant is -15: to it we add 1×25 the product of its compressor and its co-compressor, giving 10 as the first element of the new determinant. So for the next element of the same row, $5 \cdot 23 + 5 \cdot 9/5 \cdot 25 = 340$, giving the second element: and for the third $-5+(-1)\ 25 = -30$.

Example 2.

$$\begin{array}{c}\left|\begin{array}{cccccc} 1^* & \underline{0} & \underline{2} & \underline{0} & \underline{3} & \underline{0} \\ \underline{0} & 1 & 0 & 2 & 0 & 1 \\ \underline{1} & 0 & 2 & 0 & 1 & 0 \\ \underline{0} & 1 & 0 & 3 & 0 & 1 \\ \underline{2} & 0 & 1 & 0 & 3 & 0 \\ \underline{0} & 3 & 0 & 4 & 0 & 1 \end{array}\right| \\ \begin{array}{cccccc} & 0 & -2 & 0 & -3 & 0 \end{array}\end{array} = \left|\begin{array}{ccccc} 1 & 0 & 2 & \underline{0} & 1 \\ \underline{0} & \underline{0} & \underline{0} & -2^* & \underline{0} \\ 1 & 0 & 3 & \underline{0} & 1 \\ 0 & -3 & 0 & \underline{-3} & 0 \\ 3 & 0 & 4 & \underline{0} & 1 \end{array}\right| = -2 \left|\begin{array}{cccc} 1 & \underline{0} & 2 & 1 \\ 1 & \underline{0} & 3 & 1 \\ \underline{0} & -3^* & \underline{0} & \underline{0} \\ 3 & \underline{0} & 4 & 1 \end{array}\right|$$

$$= -6 \begin{array}{c}\left|\begin{array}{ccc} 1^* & \underline{2} & \underline{1} \\ \underline{1} & 3 & 1 \\ \underline{3} & 4 & 1 \end{array}\right| \\ \begin{array}{ccc} & -2 & -1 \end{array}\end{array} = -6 \left|\begin{array}{cc} 1 & 0 \\ -2 & -2 \end{array}\right| = -6\ (-2) = 12.$$

The diagonal from -2, which was chosen as chief element of the second stage above, strikes an odd marginal place, and hence its sign does not change. The diagonal from -3, the chief element of the third stage, strikes an even place, and therefore the sign changes when -3 becomes an external factor.

PRINCETON UNIVERSITY, SEPTEMBER, 1899.

THE DEVELOPMENT OF FUNCTIONS.

By S. A. Corey.

The development of functions into series has been effected to a large extent, as is well known, by the use of Taylor's, Maclaurin's and Lagrange's formulas. There are, however, a large number of cases where these formulas are of no practical value in determining the function sought. In some cases the resulting series, while theoretically convergent, become practically useless on account of the complicated character of the successive derivatives of the function. In many other important cases the resulting series do not converge, or converge too slowly to be of service for numerical computation.

It is the purpose of this paper to call attention to other formulas, similar in some respects to the older formulas, but more rapidly convergent. As far as I am aware these formulas are new. If I am in error on this point I should be glad to be so informed.

Using the notation ordinarily employed in Taylor's and Maclaurin's formulas, these new formulas may be written:

$$f(a+x)=f(a)+\frac{x}{2}[f'(a)+f'(a+x)]+\frac{x^2}{2^2\,2\,!}[f''(a)-f''(a+x)]$$
$$+\frac{x^3}{2^3\,3\,!}[f'''(a)+f'''(a+x)]+\frac{x^4}{2^4\,4\,!}[f^{(4)}(a)-f^{(4)}(a+x)]$$
$$+\frac{x^5}{2^5\,5\,!}[f^{(5)}(a)+f^{(5)}(a+x)]+\cdot\cdot\cdot\cdot\cdot\cdot\,. \qquad [I]$$

$$f(a+2x)=f(a)+2\left[\frac{x}{1}f'(a+x)+\frac{x^3}{3\,!}f'''(a+x)+\frac{x^5}{5\,!}f^{(5)}(a+x)+\cdot\cdot\cdot\cdot\right]. \qquad [II]$$

$$f(a+x)=f(a)+\frac{x}{2m}\Big\{f'(a)+f'(a+x)+2\Big[f'\Big(a+\frac{x}{m}\Big)+f'\Big(a+\frac{2x}{m}\Big)+\cdot\cdot\cdot\cdot$$
$$+f'\Big(a+\frac{(m-1)x}{m}\Big)\Big]\Big\}$$
$$+\frac{x^2}{2^2 2\,!m^2}\Big[f''(a)-f''(a+x)\Big]+\frac{x^3}{2^3 3\,!m^3}\Big\{f'''(a)+f'''(a+x)$$
$$+2\Big[f'''\Big(a+\frac{x}{m}\Big)+f'''\Big(a+\frac{2x}{m}\Big)+\cdot\cdot\cdot+f'''\Big(a+\frac{(m-1)x}{m}\Big)\Big]\Big\}$$
$$+\frac{x^4}{2^4 4\,!m^4}\Big[f^{(4)}(a)-f^{(4)}(a+x)\Big]+\cdot\cdot\cdot\cdot\cdot \qquad [III]$$

$$f(a+2mx)=f(a)+\frac{2x}{1}\Big\{f'(a+x)+f'(a+3x)+\cdots\cdot+f'[a+(2m-1)x]\Big\}$$
$$+\frac{2x^3}{3!}\Big\{f'''(a+x)+f'''(a+3x)+\cdots\cdot+f'''[a+(2m-1)x]\Big\}$$
$$+\frac{2x^5}{5!}\Big\{f^{(5)}(a+x)+f^{(5)}(a+3x)+\cdots$$
$$+f^{(5)}[a+(2m-1)x]\Big\}+\cdots \qquad \text{[IV]}$$

Formula (I) can be verified by developing $f'(a+x), f''(a+x)$, etc., in the second member into power series by Taylor's Theorem and combining terms including like powers of x. It at once reduces to the familiar development of $f(a+x)$.

If both members of formula (II) are differentiated with respect to x there will result the Taylor's development of $f'[(a+x)+x]$.

Formula (III) may be inferred from (I) without much labor by a step to step process, *i. e.* by developing by (I):

$$f\left(a+\frac{x}{m}\right), f\left[\left(a+\frac{x}{m}\right)+\frac{x}{m}\right], f\left[\left(a+\frac{2x}{m}\right)+\frac{x}{m}\right]\text{etc.},$$

simplifying each by the aid of the preceding.

Formula (IV) may be deduced by a similar step by step process from (II).

It is to be observed that the developments given by (I), (II), (III), and (IV) are not power series, and while they are perfectly trustworthy for purposes of computation whenever the Taylor's development, on which they ultimately depend, is valid, it is by no means clear that they may be safely used for other purposes; for instance that their term by term integrals or derivatives will be equal to the integral or derivative of the function.

That formulas (I) and (II) converge more rapidly than Taylor's Formula is easily seen by comparing the general term in each with the corresponding term of Taylor's Formula. The general term of (I) is

$$\frac{x^n}{2^n n!}\Big[f^{(n)}(a)-(-1)^n f^{(n)}(a+x)\Big],$$

ot Taylor's Formula is $\frac{x^n}{n!}f^{(n)}(a)$.

The general term of (II) is $\left[1-(-1)^n\right]\frac{x}{n!}f^{(n)}(a+x)$,

the corresponding term of Taylor's Formula is $\frac{2^n x^n}{n!}f^{(n)}(a)$.

Formulas (III) and (IV) can be made to converge as rapidly as may be desired by taking a sufficiently large value of m, and are especially valuable when the labor of calculating the successive derivatives of the function increases rapidly with the increase in the order of the derivative.

The following are some of the developments obtained by formulas (I) and (II):

$$(a+x)^n = a^n + \frac{n\left[a^{n-1} + (a+x)^{n-1}\right]}{2}x + \frac{n(n-1)\left[a^{n-2}-(a+x)^{n-2}\right]}{8}x^2$$
$$+\frac{n(n-1)(n-2)\left[a^{n-3}+(a+x)^{n-3}\right]}{2^3\, 3!}x^3 + \cdots\cdots \tag{1}$$

If n is a positive whole number and odd the second member of (1) will terminate with the nth power of x. If n is an even whole number and positive the second member will terminate with the $(n-1)$th power of x.

If n is equal to $1/m$ we get, after reduction, the following expression for the m th root of $(a+x)$:

$$(a+x)^{\frac{1}{m}} = a^{\frac{1}{m}}\left[\frac{1+\dfrac{x}{2ma} - \dfrac{(m-1)x^2}{8m^2a^2} + \dfrac{(1-3m+2m^2)}{48m^3a^3}x^3 - \cdots\cdots}{1-\dfrac{x}{2ma} - \dfrac{(m-1)x^2}{8m^2a^2} - \dfrac{(1-3m+2m^2)}{48m^3a^3}x^3 - \cdots\cdots}\right]. \tag{2}$$

If, as a special case, we let $a=32$, $x=-2$, and $m=5$, we get (by substituting in the foregoing expression) 1.97435047 as the fifth root of 30.

Expanding $\log(1+x)$ by formula (I) we have:

$$\log(1+x) = \frac{1+\dfrac{1}{1+x}}{2}x - \frac{1-\dfrac{1}{(1+x)^2}}{8}x^2 + \frac{2+\dfrac{2}{(1+x)^3}}{48}x^3 - \cdots\cdots \tag{3}$$

Expanding by formula (II):

$$\log(1+x) = 2\left[\frac{x}{2+x} + \frac{x^3}{3(2+x)^3} + \frac{x^5}{5(2+x)^5} + \cdots\cdot\right]. \tag{4}$$

It may be noted that (4) is valid for any value of x.

Expanding e^x by formula (I) and reducing we have:

$$e^x = \frac{1 + \frac{x}{2} + \frac{x^2}{8} + \frac{x^3}{48} + \frac{x^4}{2^4 4\,!} + \cdots\cdots}{1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{48} - \frac{x^4}{2^4 4\,!} - \cdots\cdots}. \tag{5}$$

Expanding e^{2x} by formula (II):

$$e^{2x} = 1 + 2\,e^x \left(x + \frac{x^3}{3\,!} + \frac{x^5}{5\,!} + \frac{x^7}{7\,!} + \cdots\cdots \right). \tag{6}$$

Expanding $\sin x$ by formula (I) and reducing:

$$\sin x = \frac{\cos x \left(\frac{x}{2} - \frac{x^3}{48} + \frac{x^5}{2^5 5\,!} - \cdots\cdots \right) + \left(\frac{x}{2} - \frac{x^3}{48} + \frac{x^5}{2^5 5\,!} - \cdots\cdots \right)}{1 - \frac{x^2}{8} + \frac{x^4}{2^4 4\,!} - \frac{x^6}{2^6 6\,!} + \cdots\cdots}. \tag{7}$$

Elliptic and hyperelliptic integrals and a large class of Abelian integrals may be represented by:

$$\int \frac{dx}{(X)^r}$$

where X is some polynomial in x, and r some fraction. In this class of functions the successive derivatives are usually very complicated, and formula (III) or formula (IV) will frequently be useful.

I will give but one example, namely, the evaluation of the well-known elliptic integral,

$$\int_0^x \frac{dx}{[(1-k^2x^2)(1-x^2)]^{\frac{1}{2}}}, \text{ where } x = \sqrt{\tfrac{1}{2}}, \text{ and } k = \sqrt{\tfrac{1}{2}}.$$

Using but two terms of formula (III), and making m equal to 10, we get .825+ as the value of the integral; using four terms with the same value of m, we get .82605, accurate to the last place of decimals.

Formula (IV) might have been employed in this case with equal accuracy.

In a process of approximation, such as that described by W. E. Durand, in THE ANNALS OF MATHEMATICS, 1st Series, Vol. 12, p. 110, formulas (I), (II), (III), and (IV) can usually be employed with decided advantage.

Cases might be enumerated indefinitely to show the developments obtainable by the formulas herein given, and to show the rapid convergence of the resulting series in the majority of cases, but such enumeration of cases, while interesting and instructive, would unduly lengthen this paper. Enough has been said, I believe, to arouse the interest of working mathematicians in the formulas given and to induce them to further study and develop the methods herein employed.

HITEMAN, IOWA, OCTOBER, 1899.

ILLUSTRATION OF THE ELLIPTIC INTEGRAL OF THE FIRST KIND BY A CERTAIN LINK-WORK.

By Arnold Emch.

1. The element of the link-work consists of a cell formed by six bars OA_1, A_1B_1, B_1A_2, A_2O, QB_1, and OQ, fig. 1. The first four are of equal

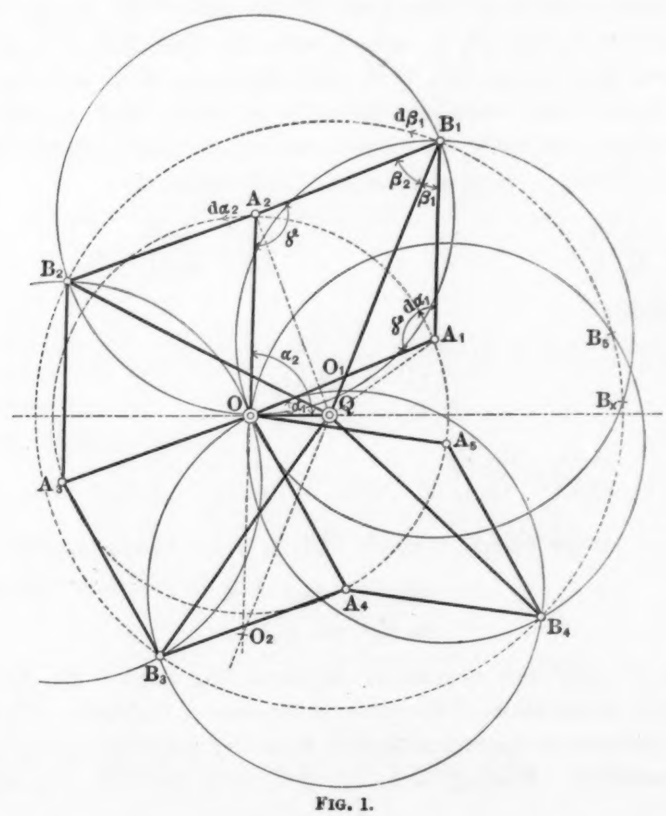

Fig. 1.

length and form a rhombus whose only fixed point is O, while QB_1 is of different length and movable about the fixed point Q. The bar OQ is fixed. As this link-work consists of 5 joints and 6 bars it is movable and has one

degree of freedom $((2 \times 5 - 3) - 6 = 1)$.* The motion is unlimited, *i. e.*, the cell can make complete revolutions, if

$$OA_1 + A_1B_1 > OQ + QB_1.$$

It is limited if

$$OA_1 + A_1B_1 < OQ + QB_1.$$

I shall consider the motion of the cell in the first case, where it can make complete revolutions.

2. Let δa_1, δa_2, $\delta\beta_1$ be the infinitesimal displacements of the points A_1, A_2, B_1 in a virtual displacement of the cell; a_1, a_2 the angles which the links OA_1, OA_2 include with the positive part of the axis OQ; β_1, β_2 the angles which the links A_1B_1 and B_1A_2 include with the link QB_1; O_1 and O_2 the points of intersection of the link QB_1 with the links OA_1 and OA_2 respectively; and, finally, the variable distances $\rho_1 = QA_1$ and $\rho_2 = QA_2$. The points O_1 and O_2 are evidently the virtual centres of rotation of the links A_1B_1 and B_1A_2 respectively. Hence, from fig. 1, the relations:

$$\frac{\delta a_1}{\delta\beta_1} = \frac{A_1O_1}{B_1O_1}, \qquad (1) \qquad\qquad \frac{\delta a_2}{\delta\beta_2} = \frac{A_2O_2}{B_1O_2}, \qquad (2)$$

from which follows

$$\frac{\delta a_1}{\frac{A_1O_1}{B_1O_1}} = \frac{\delta a_2}{\frac{A_2O_2}{B_1O_2}}. \qquad (3)$$

Now $\qquad \dfrac{A_1O_1}{B_1O_1} = \dfrac{\sin\beta_1}{\sin\gamma}, \qquad \dfrac{A_2O_2}{B_1O_2} = \dfrac{\sin\beta_2}{\sin\gamma},$

where $\qquad$ *angle* $OA_1B_1 =$ *angle* $OA_2B_1 = \gamma$. Consequently,

$$\frac{\delta a_1}{\sin\beta_1} = \frac{\delta a_2}{\sin\beta_2}. \qquad (4)$$

As there is only one degree of freedom, the angles β_1, β_2, a_1, and a_2 may be regarded as functions of the same independent variable. This differential equation assumes a more intelligible form by introducing $QA_1 = \rho_1$ and $QA_2 = \rho_2$ as variables. Putting $OA_1 = r$, $QB_1 = R$, and $OQ = e$, we have:

$$\cos a_1 = \frac{r^2 + e^2 - \rho_1^2}{2re},$$

---

* See Cremona, *Graphic Statics*, p. 152, and F. Reuleaux, *Kinematics of Machinery* pp. 283-294.

and by differentiation

$$\sin \alpha_1 \cdot d\alpha_1 = \frac{\rho_1 \cdot d\rho_1}{re}.$$

But
$$\sin \alpha_1 = \frac{2}{re}\sqrt{s(s-r)(s-e)(s-\rho_1)}\,,$$

where $s = \dfrac{r+e+\rho^1}{2}$, or

$$\sin \alpha_1 = \frac{\sqrt{-[\rho_1^2-(r+e)^2][\rho_1^2-(r-e)^2]}}{2\,re}\,,$$

hence
$$d\alpha_1 = \frac{2\rho_1 \cdot d\rho_1}{\sqrt{-[\rho_1^2-(r+e)^2][\rho_1^2-(r-e)^2]}}.$$

In the triangle A_1B_1Q

$$\sin \beta_1 = \frac{\sqrt{-[\rho_1^2-(R+r)^2][\rho_1^2-(R-r)^2]}}{2\,Rr},$$

so that

$$\frac{d\alpha_1}{\sin \beta_1} = \frac{4Rr\rho_1 \cdot d\,\rho_1}{\sqrt{[\rho_1^2-(r+e)^2][\rho_1^2-(r-e)^2][\rho_1^2-(R+r)^2][\rho_1^2-(R-r)^2]}}.$$

To abbreviate let $R+r=a$, $r+e=b$, $r-e=c$, $R-r=d$, $\rho_1^2=x$, $\rho_1 \cdot d\rho_1 = \frac{1}{2}\,dx$, so that finally

$$\frac{d\alpha_1}{\sin \beta_1} = \frac{2Rr \cdot dx}{\sqrt{(x-a)(x-b)(x-c)(x-d)}}. \qquad (5)$$

In a similar manner, if $QA_2 = \rho_2$, and $\rho_2^2 = y$,

$$\frac{d\alpha_2}{\sin \beta_2} = \frac{2Rr \cdot dy}{\sqrt{(y-a)(y-b)(y-c)(y-d)}}. \qquad (6)$$

Putting
$$\int_c^x \frac{dx}{\sqrt{(x-a)(x-b)(x-c)(x-d)}} = u\,, \qquad (7)$$

$$\int_c^y \frac{dy}{\sqrt{(y-a)(y-b)(y-c)(y-d)}} = v^*\,, \qquad (8)$$

* As c is the smallest real value of x or y, we assumed it as the lower limit. The largest real value of x or y, b, might also be taken as the lower limit.

according to equation (4), we have:

$$v - u = h \text{ (constant)}. \tag{9}$$

By inversion of the elliptic integrals (7) and (8) the elliptic functions

$$x = \lambda(u) \quad , \; y = \lambda(v) \tag{10}$$

are obtained. In this manner the cosines of the angles α_1 and α_2 may be rationally expressed by elliptic functions, and it is found that *the difference of the arguments belonging to these angles is constant and independent of the position of the cell.*

3. As indicated in fig. 1, other equal cells $(OA_2B_2A_3 \cdot B_2Q)$, $(OA_3B_3A_4 \cdot B_3Q)$ . . . may be added to the first, which together form a general link-work. In this process of adding cells two principal cases may occur: (1) the link-work will close after a certain number of additions of cells, *i. e.*, the last point A obtained in the construction will coincide with the first of the points A; (2) the link-work does not close.

To discuss the conditions of a closed link-work assume that there are n cells in it, so that the point A_{n+1} of the nth cell $OA_nB_nA_{n+1} \cdot QB_n$ will coincide with the first point A_1. The argument belonging to the angle α_1 or the point A_1 being u, the argument of A_2 will be $u+h$, of A_3 $u+2h$, . . ., of A_{n+1} $u+nh$. But A_{n+1} coincides with A_1, hence, designating the periods of the elliptic function $\lambda(u)$ by w_1 and w_2,

$$u + nh \equiv u \pmod{w_1, w_2}.$$

This condition is satisfied if

$$h \equiv 0 \ \left(\text{mod} \frac{w_1}{n}, \frac{w_2}{n}\right),$$

or

$$h = \frac{m_1 w_1 + m_2 w_2}{n}, \tag{11}$$

where m_1 and m_2 designate integers. Consequently the problem of a closed link-work is solved if h is given one of the values contained in (11). This condition necessarily requires a special arrangement of the link-work; but it does not assign any particular value to the argument u. Thus, the first point A_1 of the link-work may be chosen anywhere on the circle having O as a centre and OA_1 as a radius; the link-work closes every time and contains n cells. This result may be stated in the theorem:

If a link-work of the prescribed kind based upon two fixed circles (one having O as a centre and r as a radius, the other Q as a centre and R as a radius) closes and contains n cells, every other link-work, based upon the same two circles, closes and contains n cells.

4. In order to reduce the integral

$$\int \frac{dx}{\sqrt{(x-a)(x-b)(x-c)(x-d)}}$$

to Legendre's normal form, we have to notice that in the case of an unlimited motion $2r > R + e$, or $(r-e) > (R-r)$, or $(r-e)^2 > (R-r)^2$. But we have also $R + r > r + e$, and $r + e > r - e$, hence $R + r > r + e > r - e > R - r$, or

$$a > b > c > d. \tag{12}$$

In our case we always have $b > x > c$, so that according to a well-known formula*

$$\int_c^x \frac{dx}{\sqrt{(x-a)(x-b)(x-c)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} \operatorname{sn}^{-1} \sqrt{\frac{(b-d)(x-c)}{(b-c)(x-d)}}, \tag{13}$$

with the modulus $\kappa^2 = k = \dfrac{(b-c)(a-d)}{(a-c)(b-d)}$.

Putting this integral, as in formula (5), equal to u, we have:

$$\sqrt{\frac{(b-d)(x-c)}{(b-c)(x-d)}} = \operatorname{sn}\left(\frac{\sqrt{(a-c)(b-d)}}{2} \cdot u\right), \tag{14}$$

or, putting $\dfrac{\sqrt{(a-c)(b-d)}}{2} \cdot u = w$,

$$\frac{(x-c)(b-d)}{(x-d)(b-c)} = \operatorname{sn}^2 w. \tag{15}$$

From this

$$x = \frac{c(b-d) - d\,(b-c)\operatorname{sn}^2 w}{(b-d) - (b-c)\operatorname{sn}^2 w}. \tag{16}$$

For $u = 0$, $x = c = (r-e)^2$. The corresponding value of y is easily found as

$$y = re + \frac{r^3 - eR^2}{r-e} = p. \tag{17}$$

* See Greenhill, *Elliptic Functions*, pp. 53-55.

This value of y belongs to the argument $v=h$, since $v-u=h$; hence the constant h is determined by

$$re+\frac{r^3-eR^2}{r-e}=\frac{c(b-d)-d(b-c)\ \mathrm{sn}^2\left(\frac{\sqrt{(a-c)(b-d)}}{2}\cdot a\right)}{b-d\ -\ (b-c)\ \mathrm{sn}^2\left(\frac{\sqrt{(a-c)(b-d)}}{2}\cdot a\right)}. \tag{18}$$

Designating the real half-period of sn w by $2K$, we have:

$$\mathrm{sn}\ (w+2K)=-\ \mathrm{sn}\ w,$$

or

$$\mathrm{sn}^2\ (w+2K)=\mathrm{sn}^2\ w,$$

i. e., $2K$ is the real period of $\mathrm{sn}^2 w$. For $w=0$, $\mathrm{sn}^2 w=0$ and $x=(r-e)^2$. For $w=K$, $\mathrm{sn}^2 w=1$ and $x=b=(r+e)^2$. For $w=2K$, $\mathrm{sn}^2 w=0$ and $x=c=(r-e)^2$. To find the corresponding value of x, belonging to $w=K/2$, we make use of the formula:*

$$\mathrm{sn}\frac{K}{2}=\frac{1}{\sqrt{1+\sqrt{k'}}}, \tag{19}$$

where

$$k'=\frac{(a-b)(c-d)}{(a-c)(b-d)}$$

is the complementary modulus. Thus, for $w=K/2$, from formula (16) we obtain

$$x=\frac{b(c-d)+c(b-d)\sqrt{k'}}{(c-d)+(b-d)\sqrt{k'}}. \tag{20}$$

5. Example of 3 Cells. As the period of $\mathrm{sn}^2 w$ is $2K$, we have to put $w=2K$, in order to obtain the relation of R, r, e, in this particular closed link-work. Designating sn $w/3$ simply by S, we have:

$$\mathrm{sn}\ w=\frac{3S-4(1+k)S^3+6kS^5-k^2S^9}{1\ -6kS^4+4(1+k)kS^6-3k^2S^8}, \tag{21}$$

and since sn $2K=0$, the condition becomes

$$k^2S^8-6kS^4+4(1+k)\ S^2-3=0. \tag{22}$$

* For the formulas used and developed here and in the next two sections we refer to Greenhill, *loc. cit.*, pp. 120-121.

According to formulas (17) and (15):

$$S^2 = \frac{(x-c)(b-d)}{(x-d)(b-c)}.$$

Designating this expression by q, the required condition is

$$k^2q^4 - 6\,kq^2 + 4\,(1+k)\,q - 3 = 0. \tag{23}$$

Substituting in this expression the values of k and q in terms of R, r, e, it is easily found that condition (23) reduces to

$$R = r. \tag{24}$$

Thus, the three cell link-work is completely determined by fixing r and e. In fig. 2, $OQ = e$ and $OA_1 = r$. Now $R = r$, hence, in this case, $A_1B_1 = QB_1$. Having fixed the point B_1 it is an easy matter to complete the construction of

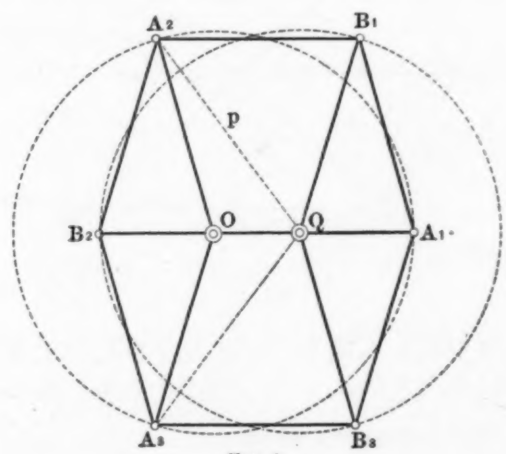

FIG. 2.

the cells $A_1B_1A_2O$, $OA_2B_2A_3$, $OA_3B_4A_1$. It is seen that $QB_1 = QB_2 = QB_4$, so that also $QB_2A_2B_1$, $QB_1A_1B_4$, $QB_4A_3B_2$ may be considered as cells of the link-work. The points O, A_1, A_2, A_3 may be interchanged with the points Q, B_1, B_2, B_3 without changing the character of the link-work.

6. Example of 4 Cells. In this case the value of y as given by formula (17) is also equal to the value of x in formula (20), *i. e.*,

$$p = \frac{b(c-d) + c(b-d)\sqrt{k'}}{(c-d) + (b-d)\sqrt{k'}}.$$

Substituting in this equation for a, b, c, d, p, and k' their values in terms of R, r, e, the condition between R, r, and e is found:

$$2r^2 = R^2 + e^2. \tag{25}$$

In this case the value of $x = p$ is

$$p = r^2 - e^2 = R - r^2,$$

as is also seen from fig. 3, in which $x = \overline{QA_2}^2 = \overline{OA_2}^2 - \overline{OQ}^2 = r^2 - e^2$, and also $x = \overline{QB_2}^2 - \overline{A_2B_2}^2 = R^2 - r^2$. From this figure it is apparent that during the motion the following groups of parallel links are maintained:

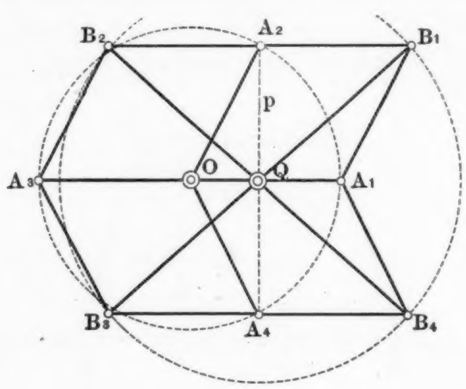

FIG. 3.

$$A_1B_1 \parallel OA_2 \parallel A_3B_2\,, \qquad A_1B_4 \parallel OA_4 \parallel A_3B_3\,,$$
$$B_1A_2 \parallel OA_1 \parallel B_4A_4\,, \qquad B_2A_2 \parallel OA_3 \parallel B_3A_4\,.$$

It follows from this that during the motion

$$B_1B_4 = A_2A_4 = BB_3,$$

and

$$B_1B_2 = A_1A_3 = B_4B_3.$$

Consequently, the points $B_1B_2B_3B_4$ always form a parallelogram, in which

$$QB_1 = QB_2 = QB_3 = QB_4.$$

But $\quad B_1B_3 = QB_1 + QB_3\,,$

and $\quad B_2B_4 = QB_2 + QB_4\,,$

hence $\quad B_1B_3 = B_2B_4.$

The parallelogram has, therefore, equal diagonals, and is a rectangle. The

closed link-work is, consequently, also completely determined by connecting the points B_1 and B_3, and B_2 and B_4 by links of equal length, and assuming

$$2r=\sqrt{2(R^2+e^2)}>R+e\ ,$$

where e is any real quantity satisfying the implied condition.

These two links always cross each other at a point Q which does not change its distance from O during the motion.

7. The Open Link-Work. Consider a link-work of the prescribed kind which does not close or which is not completed so as to form a closed link-work. Suppose there are m cells in the link-work, and that the last cell does not overlap the first.* In this manner an angle $A_{m+1}OA_1$ is formed between the last and first cell. This angle, which will be designated by ϕ, is variable during the motion, and can be expressed by elliptic functions, for,

$$\phi=\alpha_{m+1}-\alpha_1 \tag{26}$$

is a function of the argument u.

The condition for a maximum or minimum of the angle ϕ is

$$\frac{d\phi}{du}=\frac{d\alpha_{m+1}}{dx_{m+1}}\cdot\frac{dx_{m+1}}{du}-\frac{d\alpha_1}{du_1}\cdot\frac{dx_1}{du}=0\ . \tag{27}$$

According to previous formulas

$$\frac{d\alpha}{dx}=\frac{1}{\sqrt{-(x-b)(x-c)}}\ ,\ \text{and}\ \frac{dx}{du}=\sqrt{(x-a)\,(x-b)\,(x-c)\,(x-d)}.$$

Substituting these expressions, with the proper indices, in (27), the condition reduces to

$$(x_1-a)(x_1-d)=(x_{m+1}-a)(x_{m+1}-d)\ ,$$

or

$$x_1^2-x_{m+1}^2=(a+d)(x_1-x_{m+1})\ . \tag{28}$$

This equation is satisfied in two ways:

(1) when $x_1=x_{m+1}$, (29)

(2) when $x_1+x_{m+1}=a+d=2\ (R^2+r^2)$. (30)

In the first case the condition $x_1=x_{m+1}$ does not assign any relation between R, r, and e and holds therefore for every proper link-work.

Considering a complete revolution of a link-work, fig. 1, it can easily be proved that there are only two positions of the link-work possible where

* This assumption is made in order to have a clearer idea of the link-work, although the results hold also in the most general case.

$x_1 = x_{m+1}$. This is the case every time that the cell has a symmetrical position with regard to the axis OQ, which, in these cases, bisects the open space of the link-work. Suppose now that the link-work makes a complete revolution, starting from the position of the maximum angle. The angle cannot pass through zero, because the system would then be permanently closed, so that there must be a minimum between the two maxima. Similarly there must be a maximum between two minima. This result may be summed up in the theorem:

The angle formed by an open link-work can assume only one maximum and one minimum during a complete revolution.

The maximum and minimum angles are both bisected by the diameter OQ.

If the angle becomes zero, it will remain zero. In this case we still have $x_1 = x_{m+1}$ (coincident); but for every position of the link-work. Thus, we see that the case of a closed link-work is included in case (1). The second condition $x_1 + x_{m+1} = 2\ (R^2 + r^2)$ can only be satisfied in a singular case, since $x_1 + x_{m+1}$, for all possible link-works, with constant values of R and r, may be considered as a function of m and e, having for all values of m and e a constant value. From formula (16) it appears that $x_1 + x_{m+1}$ can be independent of m and e only if $e = 0$. In this case $x_1 + x_{m+1} = 2r^2$, and, according to (30), $R = 0$. There is no proper link-work.

Without entering into mechanical details of the link-work it is interesting to mention the seemingly paradoxical fact, that all our link-works have one degree of freedom in their motion, although the closed link-work satisfies the condition of a rigid frame-work.

8. Geometrical Transformation of the Link-Work. With $A_1, A_2, A_3, \ldots$, in the previous figures, as centres and r as a radius describe a series of circles. These circles all pass through O and intersect the circle of centre Q and radius R in the points B_1, B_x; B_1, B_2; B_2, B_3; B_3, B_4; ... respectively. In a closed link-work this series of circles closes also, so that the last point of intersection B_{n+1} will coincide with the first point B_x. This result may be stated in the following form:

If two fixed circles A and B are given, a series of circles can be drawn, whose centres $A_1, A_2, A_3, \ldots$ all lie on the circle A and which all pass through the centre O of A. The first circle A_1 of this series intersects circle B in two points B_x. The second circle A_2 passes through B_1 and intersects circle B a second time in B_2. The third circle passes through B_2 and intersects B a

second time in B_3, and so forth. In this manner a series of circles is obtained which may be divided into three different classes:

I. The series is limited, *i. e.*, the construction cannot be continued indefinitely.

II. The series closes, *i. e.*, after the construction of a certain number of circles, the last point of intersection B_{n+1} will coincide with the first B_1.

III. The series is unlimited.

According to the general theorem on the link-work it follows immediately that if the series of circles closes once, it will close in all cases, no matter where the first circle of the series is drawn. If the series does not close in one case, it never will close.

9. Poncelet's Poristic Polygons and Steiner's Circular Series. The circles of the previous series all touch a circle C of centre O and radius $2r$. Applying to this series an inversion with centre O and any radius, every

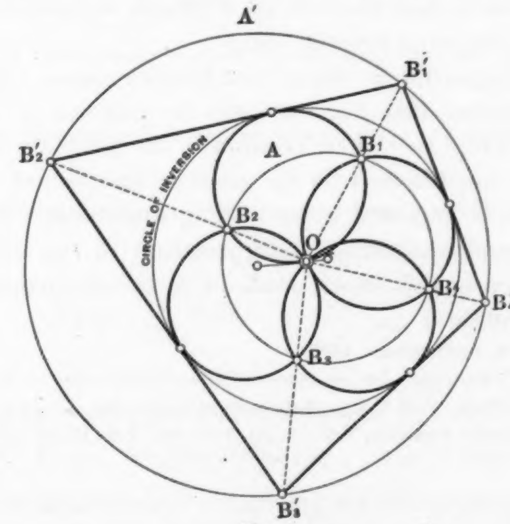

FIG. 4.

circle of the series is transformed into a straight line segment, tangent to the transformed circle of C and inscribed to the transformed circle of A. *Thus the series becomes a polygon which is inscribed to one and circumscribed to the other circle.* This is precisely the case of *Poncelet's polygons*,* fig. 4.† As

* In Poncelet's *Traité des propriétés projectives des figures* (1822) §565. See also Greenhill, *Elliptic Functions*, pp. 121–130.

† In fig. 4, C has been chosen as circle of inversion.

to the properties of closing of these polygons, it is evident that they are the same as in our link-work and the series of circles derived from it. The system of circles from which Poncelet's polygons arise may also be considered as a special case of *Steiner's circular series*,* which, in general, consists of all circles tangent to two fixed circles. From these circles a *special* series may be selected in which one point of intersection of each pair of consecutive circles always lies on a third fixed circle. These series also include the cases of Steiner's circular series where each pair of consecutive circles intersect each other under a constant angle. If this angle is zero two consecutive circles are always tangent to each other. If the first of the fixed circles of Steiner's special circular series contracts into the centre of the second fixed circle, the series arises from which Poncelet's polygons were obtained by an inversion as illustrated in fig. 4.

The properties of closing as studied by Steiner have the same character as Poncelet's polygons so that *there exists a certain equivalence between Poncelet's polygons and Steiner's circular series.*

The algebraic properties of these configurations have been studied by A. Hurwitz,† who has shown that *they rest upon the existence of more than n roots of an equation of degree n.* Their relation to the problem of the pendulum motion and Jacobi's construction for the addition theorem of elliptic functions is too well known to be repeated here. The greatest interest lies in the fact that the mechanical and geometrical interpretation of the elliptic integral of the first kind as given in this paper, leads in a simple manner to Poncelet's and Steiner's construction.

MANHATTAN, KANSAS, SEPTEMBER, 1899.

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* Steiner's *Werke*, Vol. I, pp. 19-76 and especially pp. 43-44.

† Mathematische Annalen, Vol. 15, pp. 8-15 and Vol. 19, pp. 56-66.

PROBLEMS IN THE THEORY OF CONTINUOUS GROUPS.

By Chas. L. Bouton.

1. Find the finite equations of the following G_1's, using, whenever possible, both the method of integrating a simultaneous system, and the series method:

a. $Uf = ayp + bxq$, where a and b are constants.

(Express the result in terms of hyperbolic functions.)

b. $Uf = (x^2 - y^2)\,p + 2xyq$.

c. $Uf = (x + y)^n\,(p - q)$.

d. $Uf = \rho_\mu(x_1 \cdots x_n) \sum_{i=1}^{i=n} x_i\,p_i$,

where ρ_μ is a *homogeneous* function of the x's of degree μ. Discuss the case $\mu = 0$.

2. Reduce the transformations given in 1 to canonical form (*Lie*, pp. 55, 298).*

3. Form the alternant† (Klammerausdruck) of the following transformations:

$$U_1 f = \sum_{i=1}^{i=n} x_i p_i\,, \quad U_2 f = \sum_{i=1}^{i=n} \phi_i\,p_i\,,$$

where the ϕ_i's are *homogeneous* functions, all of the *same* degree μ.

4. Form all possible new transformations by combining

$$U_1 f = p\,, \quad U_2 f = x^2 p + xyq\,, \quad U_3 f = xyp + y^2 q\,,$$

and then combining the resulting transformations.

5. Given the r infinitesimal transformations

$$U_k f = \rho_k(x_1 \cdots x_n) \sum_{i=1}^{i=n} x_i p_i\,, \quad (k = 1,\, 2,\, \cdots r),$$

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* In this article the text book, *Differentialgleichungen*, by Sophus Lie, edited by Scheffers, will be referred to as *Lie*.

† Cayley, On Reciprocants and Differential Invariants. *Quarterly Journal of Math.*, Vol. XXVI (1893), p. 303.

where the ρ_k's are homogeneous functions of the x's, *all* of the same degree μ. Show that these transformations form a group. Determine in closed form the finite equations of this G_r. Discuss the case $\mu = 0$.

6. Find the family of curves which cut the tangents of the circle $x^2 + y^2 = r^2$ at the constant angle $\tan^{-1} m$. Integrate the resulting differential equation by finding geometrically a G_1 of which it admits, and performing a quadrature. Then use the method of Scheffers (*Lie*, p. 154) to determine a second integrating factor, and so get the equation of the curves *without a quadrature*. As a special case find the equation of the involute of a circle.

Result, in the general case:

$$\tan^{-1} \frac{yr - x\sqrt{x^2 + y^2 - r^2}}{xr + y\sqrt{x^2 + y^2 - r^2}} - m \log\left[\sqrt{x^2 + y^2 - r^2} - rm\right] = \text{const.}$$

7. An ordinary differential equation of the first order admits of $U_1 f$ and $U_2 f$. If $(U_2\, U_1) \equiv 0$, show that $\Omega \equiv$ constant. (*Lie*, p. 130.)

8. Integrate the following differential equations, which define families of isothermal curves:

a. $xy'(x^2 + y^2 - 1) - y(x^2 + y^2 + 1) = 0.$

b. $xy'(x^2 - 3y^2) + y(y^2 - 3x^2) = 0.$

c. $y' \sin x \sinh y - (1 + \cos x \cosh y) = 0.$

9. Determine the types of invariant differential equations, integrating factors, and substitution to separate the variables, for the following groups, where k is a constant, and ξ, η, ϕ, ψ represent arbitrarily given functions:*

a. $Uf = kyp + xq.$

b. $Uf = kxp + yq.$

c. $Uf = \dfrac{1}{\xi(x)}\left[p + \psi'(x) q\right].$

d. $Uf = \eta(y)\left[\psi'(y)\, p + q\right].$

e. $Uf = \dfrac{1}{\phi'(x)}\, p - \dfrac{1}{\psi'(y)}\, q.$

* This list is a generalization of that given by Page, *Ordinary Differential Equations*, p. 96.

f. $Uf=\rho_\mu(x,y)\left[xp+yq\right]$,

where ρ_μ is a homogeneous function of x and y of degree μ.

g. $Uf=\dfrac{1}{\phi(x,y)}q$.

10. An m times extended transformation being

$$U^{(m)}f=\xi p+\eta q+\sum_{k=1}^{k=m}\eta_k\frac{\partial f}{\partial y^{(k)}},$$

show that

$$\eta_k\equiv\frac{d^k\eta}{dx^k}-\sum_{s=0}^{s=k-1}\frac{d^s}{dx^s}\left(y^{(k-s)}\frac{d\xi}{dx}\right),$$

where the differentiations with respect to x are total.

11. Determine all of the transformations of the following equations (*Lie*, p. 389), and hence simplify the integration of the equations as much as possible:

a. $y''-xyy'=0$.

b. $x^2y''-xy'+y=0$.

c. $xyy''-xy'^2-yy'=0$.

d. $(y-xy')y''-4y'^2=0$.*

e. $3y''y^{\text{IV}}-5y'''^2=0$.

12. Use the method of *Lie*, p. 389, to determine *all* the infinitesimal transformations which transform any circle into a circle.

13. Determine the analytic expression and the geometric meaning of the invariants of the following systems:

In the plane.

a. A right line and a circle, under the group of similitudinous transformations.

b. Two parabolas, under the general linear group.

c. A point, a right line, and a conic, under the general projective group.

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* Forsyth, *Differential Equations*, p. 125.

d. A point and two circles, under the G_6 of all circular transformations. (See 12.)

e. x, y, y', y'', and a right line, under the group of similitudinous transformations.

In space.

f. Four right lines, under the general projective group.

g. Two cones of the second degree, under the same group.

h. Two conics, under the same group.

14. Identify the group determined in 4 with one of those given by Lie, *Continuierliche Gruppen*, p. 288, and find the change of variables which carries that group into the typical one.

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CONTENTS.

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